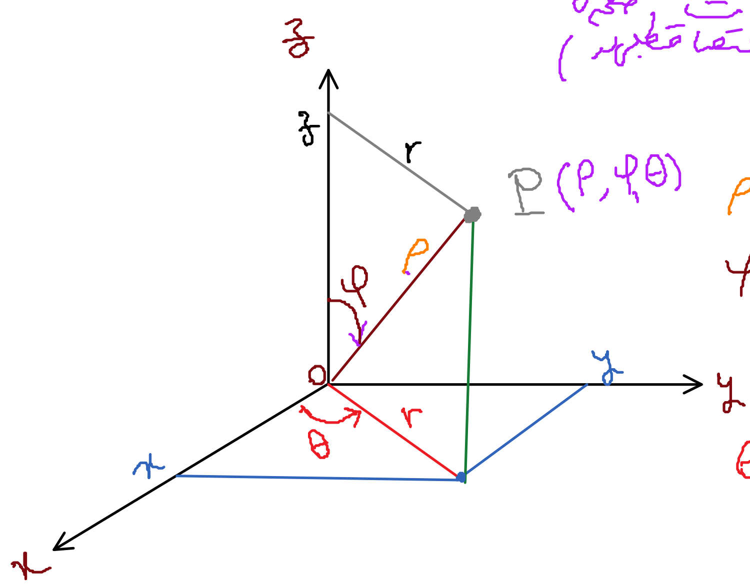
3D coordinates2

Wednesday, March 16, 2022 10:00 AM

دستگاه مختصات کروی

(دستگاه مختصات قطبی در صفحه zop + دستگاه مختصات قطبی در صفحه xy)



$\rho := |\overline{OP}|$ فاصله نقطه P از مبدأ

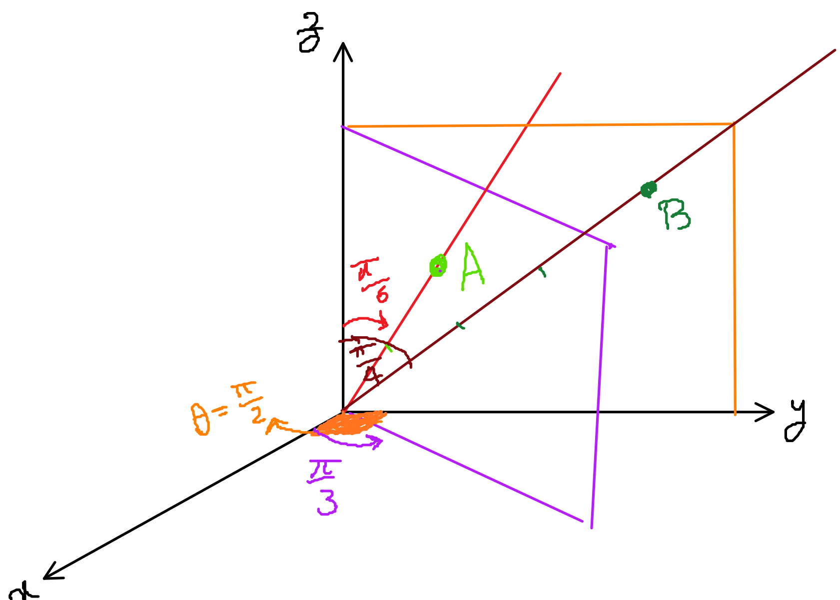
φ زاویه $\overline{OP}$ با جهت مثبت محور z

$(\varphi \in [0, \pi])$

θ زاویه قطبی

$A(2, \frac{\pi}{6}, \frac{\pi}{3})$

$B(3, \frac{\pi}{4}, \frac{\pi}{2})$



Ex. 14, P4

مکان هندسی نقاطی از فضای $\mathbb{R}^3$ را بیابید که در آن

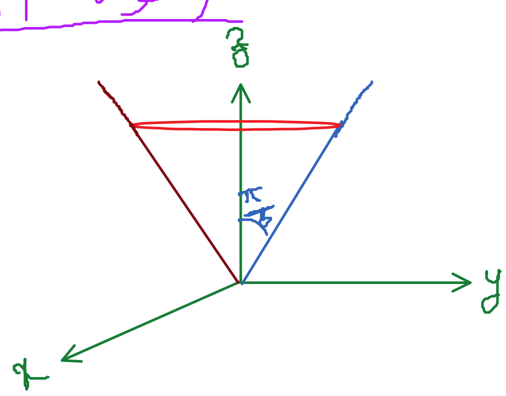
(a) $\rho = 1$

مکان هندسی نقاطی در فضا که فاصله نقطه از مبدأ مختصات برابر مقدار ثابت 1 است، کره‌ای به مرکز مبدأ و شعاع یک می‌باشد.

$\rho = 1$, (θ, φ دلخواه)

(b) $\varphi = \frac{\pi}{6}$

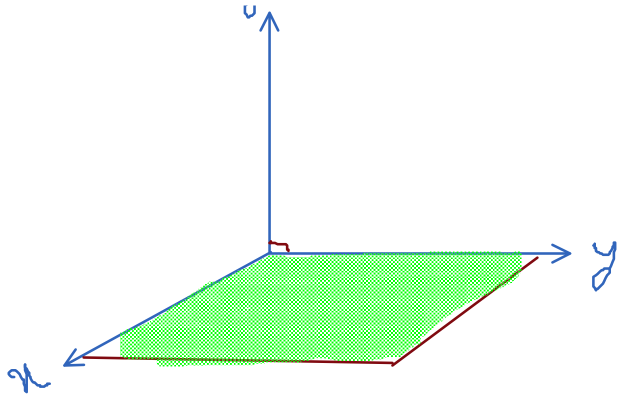
(ρ, θ دلخواه)



با توجه به اینکه ρ و θ دلخواه هستند مخروطی یکپارچه با زاویه $\frac{\pi}{6}$ داریم

(c) $\varphi = \frac{\pi}{2}$

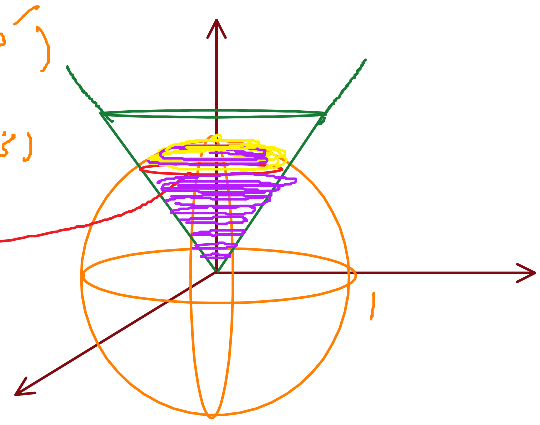
صفحه xoy (مخروط با $\frac{\pi}{2}$)



Ex15, P4

(a) $\begin{cases} \rho = 1 & \text{(کره ای به مرکز مبدأ و شعاع 1)} \\ \varphi = \frac{\pi}{6} & \text{(مخروط } \frac{\pi}{6}\text{)} \end{cases}$

دایره فصل مشترک



(b) $\begin{cases} 0 \leq \rho \leq 1 \\ \varphi = \frac{\pi}{6} \end{cases}$

آن قسمت از رویه مخروط که درون کره قرار دارد
(سطح جانبی مخروط)

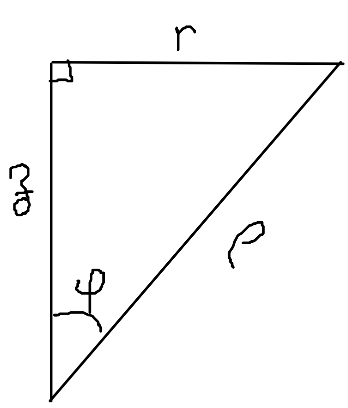
(c) $\begin{cases} \rho = 1 \\ 0 \leq \varphi \leq \frac{\pi}{6} \end{cases}$

قسمتی از سطح کره که درون قیف مخروط قرار دارد (سطح عرقچین کره)

(d) $\begin{cases} 0 \leq \rho \leq 1 \\ 0 \leq \varphi \leq \frac{\pi}{6} \end{cases}$

بستنی قیفی

روابط بین دستگاه‌ها



$$\begin{cases} r = \rho \sin\varphi \\ z = \rho \cos\varphi \\ \tan\varphi = \frac{r}{z} \\ \rho^2 = r^2 + z^2 \end{cases}$$

استوانه‌ای - کروی

$$\begin{cases} \rho^2 = r^2 + z^2 \\ \tan\varphi = \frac{r}{z} \\ \theta \quad \text{زاویه قطبی} \end{cases} \Longleftrightarrow \begin{cases} r = \rho \sin\varphi \\ \theta \quad \text{زاویه قطبی} \\ z = \rho \cos\varphi \end{cases}$$

θ زاویه قطبی $\qquad\qquad z=\rho\cos\varphi$

دکارتی - استوانه‌ای - کروی

$$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=z \end{cases} \Longleftrightarrow \begin{cases} x=\rho\sin\varphi\cos\theta \\ y=\rho\sin\varphi\sin\theta \\ z=\rho\cos\varphi \end{cases} \Longleftrightarrow \begin{cases} \rho^2=x^2+y^2+z^2 \\ \tan\varphi=\frac{r}{z}=\frac{\sqrt{x^2+y^2}}{z} \\ \tan\theta=\frac{y}{x} \end{cases}$$

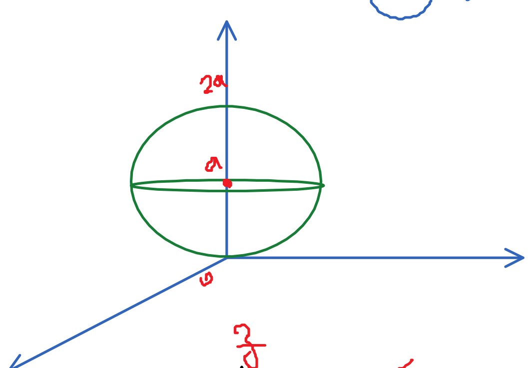
Ex. 16, P5

(a) $\begin{cases} \rho=2a\cos\varphi \\ \theta=\frac{\pi}{2} \end{cases}$ $\qquad$ (b) $\rho=2a\cos\varphi$

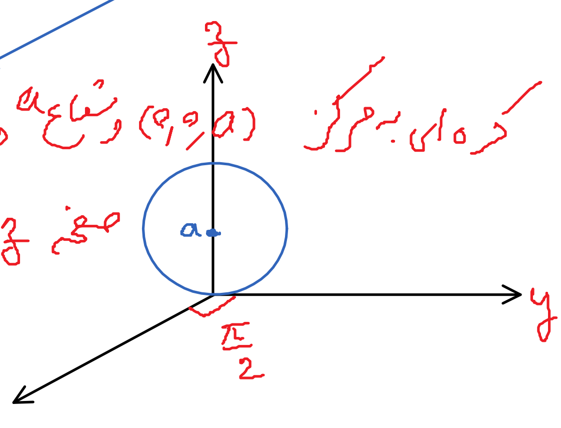
(b) $\rho=2a\cos\varphi \Rightarrow \rho^2=2a\rho\cos\varphi=2az$

$x^2+y^2+z^2-2az+a^2=a^2$

$x^2+y^2+(z-a)^2=a^2$ $\qquad$ کره‌ای به مرکز $(0,0,a)$ و شعاع a



(a) $\begin{cases} \rho=2a\cos\varphi \longrightarrow \text{کره‌ای به مرکز } (0,0,a) \text{ و شعاع } a \\ \theta=\frac{\pi}{2} \longrightarrow yz \text{ صفحه} \end{cases}$



دایره

Ex. P6

$z=\sqrt{x^2+y^2}=r$ $\qquad$ ($\frac{\pi}{4}$ مخروط با)

$\rho\cos\varphi=\rho\sin\varphi \Rightarrow \tan\varphi=1 \Rightarrow \varphi=\frac{\pi}{4}$

Vector Functions1

Monday, April 4, 2022 1:49 PM

Ex. 2, p15 $\vec{r}(t) = (a\cos\omega t, a\sin\omega t)$

$$\vec{v}(t) = (-a\omega\sin\omega t, a\omega\cos\omega t)$$

$$\vec{a}(t) = (-a\omega^2\cos\omega t, -a\omega^2\sin\omega t)$$

Ex. 3 p15

$$\vec{r}(t) = (x, x^2) = x\vec{i} + x^2\vec{j}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + 2x\frac{dx}{dt}\vec{j} = \frac{dx}{dt}(\vec{i} + 2x\vec{j})$$

$$v(t) = |\vec{v}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(2x\frac{dx}{dt}\right)^2}$$

$$= \left|\frac{dx}{dt}\right|\sqrt{1+4x^2} = \frac{dx}{dt}\sqrt{1+4x^2}$$

$$v = 5 \Rightarrow 5 = \frac{dx}{dt}\sqrt{1+4x^2} \Rightarrow \frac{dx}{dt} = \frac{5}{\sqrt{1+4x^2}}$$

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \frac{d^2x}{dt^2}(\vec{i} + 2x\vec{j}) + \frac{dx}{dt}\left(0 + 2\frac{dx}{dt}\vec{j}\right)$$

$$= \frac{d^2x}{dt^2}(\vec{i} + 2x\vec{j}) + 2\left(\frac{dx}{dt}\right)^2\vec{j}$$

$$\left.\frac{dx}{dt}\right|_{x=1} = \frac{5}{\sqrt{1+4}} = \frac{5}{\sqrt{5}} = \sqrt{5} \Rightarrow \vec{v} = \sqrt{5}\,\vec{i} + 2\sqrt{5}\,\vec{j}$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dx}\left(\frac{5}{\sqrt{1+4x^2}}\right)\frac{dx}{dt}$$

$$= \frac{-5(8x)}{2(1+4x^2)^{3/2}}\,\frac{5}{(1+4x^2)^{1/2}} = \frac{-100x}{(1+4x^2)^2}$$

$$\frac{d^2x}{dt^2}\Big|_{x=1} = \cdots = -4$$

$$\Rightarrow \vec{a} = -4(1+2\vec{j}) + 1\cdot\vec{j} = -4\vec{i} + 2\vec{j}$$

Ex 4, P15

$$\vec{F} = (\cos t, \sin t)$$

$$\vec{r}(0) = 2\vec{i}, \quad \vec{v}(0) = v_0\vec{j} = (0, v_0)$$

$$\vec{F} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2} = (\cos t, \sin t)$$

$$\int \Rightarrow m\frac{d\vec{r}(t)}{dt} = m\vec{v}(t) = (\sin t, -\cos t) + \vec{C}_1$$

$$m(0, v_0) = m\vec{v}(0) = (0, -1) + \vec{C}_1 \Rightarrow \vec{C}_1 = (0, mv_0 + 1)$$

$$m\vec{v}(t) = (\sin t, -\cos t + mv_0 + 1)$$

$$\int \Rightarrow m\vec{r}(t) = (-\cos t, -\sin t + (mv_0+1)t) + \vec{C}_2$$

$$m\vec{r}(0) = m(2, 0) = (-1, 0) + \vec{C}_2$$

$$\Rightarrow C_2 = (2m+1, 0)$$

$$\Rightarrow m\vec{r}(t) = (-\cos t + 2m + 1, -\sin t + (mv_0+1)t)$$

Ex. 6, P17

$$v^2(t) = |\vec{v}(t)|^2 = \vec{v}(t)\cdot\vec{v}(t)$$

$$\xRightarrow{\text{مشتق}} \cancel{2}v(t)\frac{dv(t)}{dt} = \vec{a}(t)\cdot\vec{v}(t) + \vec{v}(t)\cdot\vec{a}(t) = \cancel{2}\vec{v}(t)\cdot\vec{a}(t)$$

نشتق
$$\Longrightarrow \cancel{2} v(t)\frac{dv(t)}{dt} = \vec{a}(t)\cdot\vec{V}(t) + \vec{V}(t)\cdot\vec{a}(t) = \cancel{2}\vec{V}(t)\cdot\vec{a}(t) \quad (*)$$

$$v(t) \text{ ثابت } \Longleftrightarrow \frac{dv(t)}{dt} = 0 \overset{(*)}{\Longleftrightarrow} \vec{V}(t)\cdot\vec{a}(t) = 0 \Longleftrightarrow \vec{V}(t)\perp\vec{a}(t)$$

Ex. 19, P19

$$\vec{r}(t) = (a\cos t,\ a\sin t), \quad 0\le t\le 2\pi$$

$$\vec{V}(t) = (-a\sin t,\ a\cos t)$$

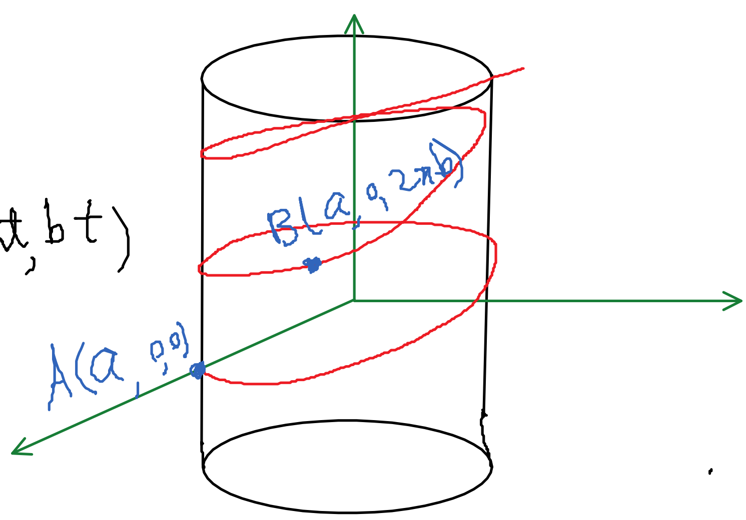
$$|\vec{V}(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a$$

$$s = \int_0^{2\pi} a\,dt = a\,t\Big|_0^{2\pi} = 2\pi a$$

Ex. 20, P19

$$\vec{r}(t) = (a\cos t,\ a\sin t,\ bt)$$

$$x^2 + y^2 = a^2$$



$$\vec{V}(t) = (-a\sin t,\ a\cos t,\ b)$$

$$|\vec{V}(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2}$$

$$A(a,0,0) \Rightarrow \begin{cases} a = a\cos t \Rightarrow \cos t = 1 \\ 0 = a\sin t \Rightarrow \sin t = 0 \Rightarrow t = 0 \\ 0 = bt \Rightarrow t = 0 \end{cases}$$

$$B(a,0,2\pi b) \Longrightarrow t = 2\pi$$

$$\int_0^{2\pi} |\vec{V}(t)|\,dt = 2\pi\sqrt{a^2+b^2}$$

$$s = \int_0^{2\pi} |\vec{v}(t)|\, dt = 2\pi\sqrt{a^2+b^2}$$

Ex. (P43) $\vec r(t) = (a\cos(\omega t), a\sin(\omega t), bt)$

$$\kappa = \left|\frac{d\vec T}{ds}\right|$$

$$\frac{d\vec T}{ds} = \frac{d\vec T}{dt}\,\frac{dt}{ds} = \frac{1}{|\vec v(t)|}\,\frac{d\vec T}{dt} \qquad (*)$$

$$\vec T(t) = \frac{\vec v(t)}{|\vec v(t)|} = \frac{(-a\omega\sin(\omega t), a\omega\cos(\omega t), b)}{\sqrt{a^2\omega^2+b^2}}$$

$$\frac{d\vec T}{dt} = \frac{1}{\sqrt{a^2\omega^2+b^2}}(-a\omega^2\cos(\omega t), -a\omega^2\sin(\omega t), 0)$$

$$\overset{(*)}{\Rightarrow} \kappa = \frac{1}{a^2\omega^2+b^2}\sqrt{a^2\omega^4\cos^2(\omega t)+a^2\omega^4\sin^2(\omega t)+0}$$

$$= \frac{a\omega^2}{a^2\omega^2+b^2}$$

$$\vec N = \frac{1}{\kappa}\,\frac{d\vec T}{ds} = \frac{1}{\kappa|v|}\,\frac{d\vec T}{dt}$$

$$= \frac{\cancel{a^2\omega^2+b^2}}{\cancel{a\omega^2}}\,\frac{1}{\cancel{\sqrt{a^2\omega^2+b^2}}}\,\frac{-\cancel{a\omega^2}}{\cancel{\sqrt{a^2\omega^2+b^2}}}(\cos\omega t, \sin\omega t, 0)$$

$$= (-\cos(\omega t), -\sin(\omega t), 0)$$

Vector Functions3

Wednesday, April 13, 2022 10:00 AM

Ex. PS3

$$\vec{r}(t) = (t+\cos t)\,\vec{i} + (t-\cos t)\,\vec{j} + \sqrt{2}\sin t\,\vec{k}$$

$$\vec{v}(t) = (1-\sin t,\ 1+\sin t,\ \sqrt{2}\cos t)$$

$$\vec{a}(t) = (-\cos t,\ \cos t,\ -\sqrt{2}\sin t)$$

$$\vec{v}\times\vec{a} = (-\sqrt{2}(1+\sin t),\ -\sqrt{2}(1-\sin t),\ 2\cos t)$$

$$|\vec{v}\times\vec{a}| = \dots = 2\sqrt{2}$$

$$|\vec{v}| = \dots = 2$$

$$\kappa = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3} = \frac{2\sqrt{2}}{2^3} = \frac{1}{2\sqrt{2}}$$

$$\tau = -\frac{d\vec{B}}{ds}\cdot\vec{N} = \frac{(\vec{v}\times\vec{a})\cdot d\vec{a}/dt}{|\vec{v}\times\vec{a}|^2}$$

$$= \dots = \frac{-2\sqrt{2}}{(2\sqrt{2})^2} = \frac{-1}{2\sqrt{2}}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{2}(1-\sin t,\ 1+\sin t,\ \sqrt{2}\cos t)$$

$$\vec{B} = \frac{\vec{v} \times \vec{a}}{|\vec{v} \times \vec{a}|} = \left(-\frac{1+\sin t}{2},\ -\frac{1-\sin t}{2},\ \frac{1}{\sqrt{2}}\cos t \right)$$

$$\vec{N} = \vec{B} \times \vec{T} = \left(-\frac{1}{\sqrt{2}}\cos t,\ \frac{1}{\sqrt{2}}\cos t,\ -\sin t \right)$$

Partial Derivatives1

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Ex. P67

(i) $f(x,y) = x^2 + 3xy + y - 1$

$$\frac{\partial f}{\partial x} = 2x + 3y + 0 - 0 = 2x + 3y$$

$$\left.\frac{\partial f}{\partial x}\right|_{(4,-5)} = 8 - 15 = -7.$$

$$\frac{\partial f}{\partial y} = 0 + 3x + 1 - 0$$

$$= 3x + 1$$

$$\frac{\partial f}{\partial y}(4,-5) = 13$$

(iii) $f(x,y) = \dfrac{2y}{y + \cos x}$

$$\frac{\partial f}{\partial x} = \frac{0 - (0 - \sin x)2y}{(y + \cos x)^2} = \frac{2y \sin x}{(y + \cos x)^2}$$

Ex. 3, P67

$$yz - \ln z = x + y$$

$$y\frac{\partial z}{\partial x} - \frac{\partial z / \partial x}{z} = 1 + 0 \Rightarrow \left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{1}{y - 1/z}$$

Ex. P68

$$z = f(x/y)$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0 \qquad \text{(PDE)}$$

$$\frac{\partial z}{\partial x} = \frac{1 \times y - 0}{y^2} f'(x/y) = \frac{1}{y} f'(x/y)$$

$$\frac{\partial f}{\partial x} = \frac{\cdots}{y^2} f'(\cdots) = \cdots$$

$$\frac{\partial f}{\partial y} = \frac{0 - x}{y^2} f'(x/y) = -\frac{x}{y^2} f'(x/y)$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \frac{x}{y} f'(x/y) - \frac{x}{y} f'(x/y) = 0$$

Ex P68

$$w = x^{y\ln z} = f(x,y,z) \qquad (e,2,e)$$

$$\frac{\partial f}{\partial x} = y\ln z\, x^{y\ln z - 1} \Rightarrow \frac{\partial f}{\partial x}(e,2,e) = 2e^{2-1} = 2e$$

$$\frac{\partial f}{\partial y}(e,2,e) \quad ?$$

$$w = x^{y\ln z} \Rightarrow \ln w = \ln x^{y\ln z} = y\ln z \ln x$$

$$\frac{\frac{\partial w}{\partial y}}{w} = 1 \times \ln z \ln x - 0 - 0 = \ln z \ln x$$

$$\Rightarrow \frac{\partial w}{\partial y}\Big|_{(e,2,e)} = \ln x \ln z\, x^{y\ln z}\Big|_{(e,2,e)} = e^2$$

$$\partial w / \partial z \quad \cdots$$

Ex. P73 $\quad w = f(x - ct) + g(x + ct) = F(x,t)$

$$\cdots \quad g'(x+ct)$$

Ex. P 7.3 $\quad w = f(x - ct) + g(x + ct)$ — (1.1)

$$\frac{\partial w}{\partial t} = -c f'(x-ct) + c g'(x+ct)$$

$$\frac{\partial w}{\partial x} = f'(x-ct) + g'(x+ct)$$

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\frac{\partial w}{\partial t} = c^2 f''(x-ct) + c^2 g''(x+ct)$$

$$\frac{\partial^2 w}{\partial x^2} = f''(x-ct) + g''(x+ct)$$

$$\Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}.$$

Partial Derivatives2

Wednesday, April 20, 2022 10:10 AM

Ex. P41

$$f(x,y) = x^2 - xy + \tfrac{1}{2}y^2 + 3$$

$$\begin{aligned} \ell(x,y) &= f(3,2) + f_x(3,2)(x-3) + f_y(3,2)(y-2) \\ &= 8 + 4(x-3) + (-1)(y-2) \\ &= 4x - y - 2 \end{aligned}$$

Ex P 84

$$f(x,y) = x^2 y$$

$D_{\vec{u}} f(-1,-1)$, $\quad \vec{u} = \left(\tfrac{1}{\sqrt{5}}, \tfrac{2}{\sqrt{5}}\right)$

$$D_{\vec{u}} f(-1,-1) = \lim_{h \to 0} \frac{f\left(-1 + \tfrac{1}{\sqrt{5}}h, -1 + \tfrac{2}{\sqrt{5}}h\right) - f(-1,-1)}{h}$$

$$= \lim_{h \to 0} \frac{\left(-1 + \tfrac{h}{\sqrt{5}}\right)^2\left(-1 + \tfrac{2h}{\sqrt{5}}\right) - (-1)^2(-1)}{h}$$

$$= \dots$$

$$= \tfrac{4}{\sqrt{5}}$$

P 84

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\lim \quad \frac{f(0+hu_1, 0+hu_2) - f(0,0)}{} \qquad \frac{hu_1 h^2 u_2^2}{h^2u_1^2 + h^4u_2^4} - 0$$

$$D_{\vec{u}} f(0,0) = \lim_{h \to 0} \frac{f(0+hu_1, 0+hu_2) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{[illegible]}{h^2u_1^2+h^4u_2^4} - 0}{h}$$

$$= \lim_{h \to 0} \frac{h^3 u_1 u_2^2}{h^3u_1^2 + h^5 u_2^4} = \lim_{h \to 0} \frac{u_1 u_2^2}{u_1^2 + h^2 u_2^4} = \frac{u_2^2}{u_1}, \quad u_1 \neq 0$$

$$u_1 = 0 \Rightarrow D_{\vec{u}} f(0,0) = \lim_{h \to 0} \frac{f(0, hu_2) - f(0,0)}{h} = \cdots = 0$$

$\vec{u} = (0, u_2)$

بنابراین، f در هر جهتی در مبدأ مشتق‌پذیر است اما به آسانی می‌توان نشان داد f در مبدأ پیوسته نیست.

Ex. p86

$$f(x,y,z) = x^3 - xy^2 - z$$

$$P_0(1,1,0)$$

$$\left.\begin{array}{l} \vec{A} = (2,-3,6) \\ |\vec{A}| = \sqrt{4+9+36} = 7 \end{array}\right\} \Rightarrow \vec{u} = \left(\frac{2}{7}, \frac{-3}{7}, \frac{6}{7}\right)$$

$$D_{\vec{u}} f(P_0) = \vec{\nabla} f(P_0) \cdot \vec{u}$$

$$\vec{\nabla} f = (3x^2 - y^2, -2xy, -1)$$

$$\vec{\nabla} f(P_0) = \vec{\nabla} f(1,1,0) = (2, -2, -1)$$

$$D_{\vec{u}} f(P_0) = (2,-2,-1)\cdot\left(\tfrac{2}{7}, \tfrac{-3}{7}, \tfrac{6}{7}\right) = \frac{4}{7}.$$

Ex. P88

$$f(x,y,z) = \frac{z}{xy}, \quad P(1,-1,2)$$

$$P \to O$$

$$\overrightarrow{PO} = (-1,1,-2), \quad |\overrightarrow{PO}| = \sqrt{6}, \quad \vec{u} = \left(\tfrac{-1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, \tfrac{-2}{\sqrt{6}}\right)$$

$$\vec{\nabla} f = (f_x, f_y, f_z) = \left(\frac{0 - yz}{x^2y^2}, \frac{0 - xz}{x^2y^2}, \frac{xy - 0}{x^2y^2}\right)$$

$$= \left(\frac{-z}{x^2 y}, \frac{-z}{x y^2}, \frac{1}{xy}\right)$$

$$\vec{\nabla} f(P) = \vec{\nabla} f(1,-1,2) = (2,-2,-1)$$

$$D_{\vec{u}} f(P) = (2,-2,-1)\cdot\left(\tfrac{-1}{\sqrt{6}}, \tfrac{2}{\sqrt{6}}, \tfrac{-2}{\sqrt{6}}\right) = -\tfrac{2}{\sqrt{6}}$$

(ب) $\vec{u}_1 = -\vec{u}$

$$D_{\vec{u}_1} f(P) = D_{-\vec{u}} f(P) = \vec{\nabla} f(P)\cdot(-\vec{u})$$

$$= -\left(\vec{\nabla} f(P)\cdot\vec{u}\right) = -D_{\vec{u}} f(P) = \tfrac{2}{\sqrt{6}}$$

(ج) $\vec{u} = \vec{\nabla} f(P) / |\vec{\nabla} f(P)|$

(د) $\vec{u} = -\vec{\nabla} f(P) / |\vec{\nabla} f(P)|$

(b) [illegible] $/\,|\vec{\nabla} f(p)|$

Ex p88

$f(x,y) = xy$

$P_0(2,0)$

$D_{\vec{u}} f(P_0) = -1$

$\vec{u}$?

$$D_{\vec{u}} f(P_0) = \vec{\nabla} f(P_0) \cdot \vec{u}$$

$$\vec{\nabla} f = (y, x)$$

$$\vec{\nabla} f(P_0) = \vec{\nabla} f(2,0) = (0,2)$$

$$\vec{u} = (u_1, u_2)$$

$$-1 = (0,2) \cdot (u_1, u_2) \Rightarrow u_2 = -1/2$$

$$|\vec{u}| = 1 \Rightarrow u_1^2 + u_2^2 = 1 \Rightarrow u_1^2 = 1 - 1/4 = 3/4$$

$$\Rightarrow u_1 = \pm \sqrt{3}/2$$

$$\vec{u} = \left(\pm \sqrt{3}/2, -1/2\right)$$

$\vec{u}$?

$$D_{\vec{u}} f(P_0) = -3 = \vec{\nabla} f(P_0) \cdot \vec{u}$$

$$= (0,2) \cdot (u_1, u_2)$$

$$\Rightarrow u_2 = -3/2$$

$$|\vec{u}| = 1 \Rightarrow u_1^2 + u_2^2 = 1 \Rightarrow u_1^2 = 1 - \frac{9}{4} = -3/4$$

چنین $\vec{u}$ای وجود ندارد.

Partial Derivatives3

Wednesday, April 27, 2022 10:03 AM

$$f(x,y)=\ln|\vec r| \Rightarrow \vec\nabla f = \frac{\vec r}{|\vec r|^2}$$
$$\vec r = x\vec i + y\vec j$$

$$f(x,y,z)=\ln|\vec r|^{-n} \Rightarrow \vec\nabla f = \frac{-n\vec r}{|\vec r|^{n+2}}$$
$$\vec r = x\vec i + y\vec j + z\vec k$$

Ex. P90

$$x^2+xyz-z^3=1 \qquad P_0(1,1,1)$$

$$\vec\nabla f=(2x+yz,\ xz,\ xy-3z^2)$$

$$\vec\nabla f(P_0)=(3,1,-2)$$

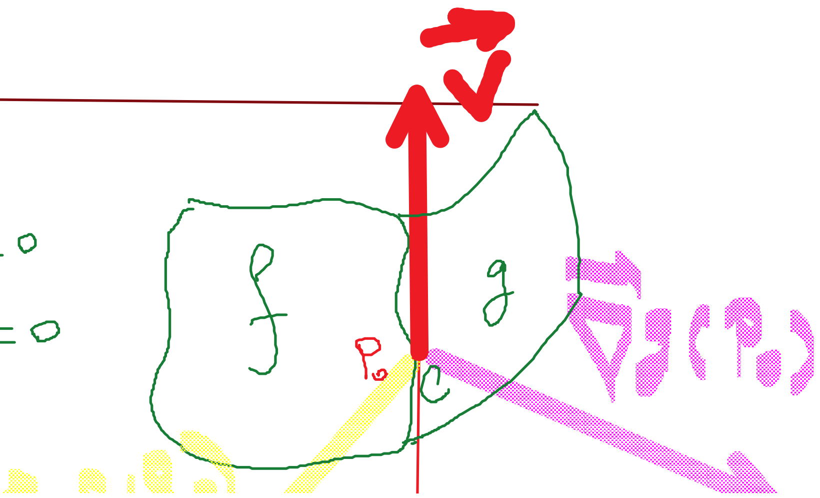
$$\vec n=(3,1,-2), \qquad \vec\ell=(3,1,-2)$$

معادله صفحه مماس: $3(x-1)+1(y-1)-2(z-1)=0$

معادله خط قائم: $\begin{cases} x=1+3t \\ y=1+t, \quad t\in\mathbb{R} \\ z=1-2t \end{cases}$

Ex2, P90

$$f(x,y,z)=x^2+y^2-z^2-1=0$$
$$g(x,y,z)=x+y+z-5=0$$
$$P_0(1,2,2)$$



$P_0(1,2,2)$

$$\vec V = \vec\nabla f(P_0) \times \vec\nabla g(P_0)$$

$$\vec\nabla f = (2x, 2y, -2z) \Rightarrow \vec\nabla f(P_0) = (2,4,-4)$$

$$\vec\nabla g = (1,1,1)$$

$$\vec V = (2,4,-4) \times (1,1,1) = (8,-6,2)$$

$$\Rightarrow \begin{cases} x = 1 + 8t, \\ y = 2 - 6t, \\ z = 2 + 2t, \end{cases} \quad t \in \mathbb{R}$$

دما در هر نقطه از ناحیه‌ای از فضا با ضابطه $T(x,y,z) = x^2 - y^2 + z^2 + xy^2$ مشخص می‌شود. یک مگس در لحظه $t=0$ از نقطه $(1,1,2)$ گذشته و در امتداد منحنی فصل مشترک سطوح $z = 3x^2 - y^2$ و $2x^2 + 2y^2 - z^2 = 0$ پرواز می‌کند. اگر تندی مگس $7 m/s$ باشد، میزان تغییر دما برای مگس در لحظه $t=0$ را محاسبه کنید.

بردار مماس $\vec V$ در مسیر پرواز در نقطه $P_0(1,1,2)$ عبارت است از:

$$\vec V = \vec\nabla f(P_0) \times \vec\nabla g(P_0)$$

$$\begin{cases} f(x,y,z) = 3x^2 - y^2 - z = 0 \end{cases} \Rightarrow \begin{cases} \vec\nabla f = (6x, -2y, -1) \\ \vec\nabla g = (4x, 4y, -2z) \end{cases}$$

$$\begin{cases} f(x,y,z) = 3x - y - z = \ldots \\ g(x,y,z) = 2x^2 + 2y^2 - z^2 = 0 \end{cases} \qquad \vec{\nabla} g = (4x, 4y, -2z)$$

$$\vec{V} = (6,-2,-1) \times (4,4,-4) = 4(3,5,8)$$

$$\vec{u} = \frac{\vec{V}}{|\vec{V}|} = \frac{(3,5,8)}{\sqrt{98}}$$

$$\vec{\nabla} T = (2x + y^2,\ -2y + 2xy,\ 2z)$$

$$\vec{\nabla} T(P_0) = (3,0,4)$$

$$7 D_{\vec{u}} T(P_0) = 7\, \vec{\nabla} T(P_0) \cdot \vec{u}$$

$$= 7\,(3,0,4) \cdot \frac{1}{\sqrt{98}}(3,5,8)$$

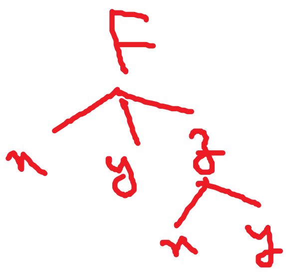
$$= \frac{287}{\sqrt{98}}.$$

مشتق گیری ضمنی

$$F(x^2 - z^2,\ y^2 + xz) = 0$$

$$z = z(x,y)$$

$$z_x \ ?$$



$$(2x - 2z z_x) F_x + (z + x z_x) F_y = 0$$

$$\vdots$$

Ex P95

$$z = 2x - x^2 + 4y - y^2 + 5$$

$$\begin{cases} z_x = 2 - 2x = 0 \\ z_y = 4 - 2y = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 2 \end{cases}$$

$$\Delta z = f(1+\Delta x, 2+\Delta y) - f(1,2)$$

$$= \cdots$$

$$= -\left[(\Delta x)^2 + (\Delta y)^2\right] \leq 0$$

$\Longrightarrow$ $(1,2)$ یک نقطه Max نسبی است.

$$z = y^2 - x^2$$

$$\begin{cases} z_x = -2x = 0 \\ z_y = 2y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

$$\Delta z = f(\Delta x, \Delta y) - f(0,0)$$

$$= (\Delta y)^2 - (\Delta x)^2$$

علامت Δz همواره ثابت نیست. در واقع، در جهت مثبت محور x تابع
f دارای مقدار $f(x,0) = -x^2 < 0$، و در جهت مثبت محور y تابع
f دارای مقدار $f(0,y) = y^2 > 0$ است. لذا هر گوی باز
در صفحه xoy حول مبدأ شامل نقاطی است که تابع

در صفحه xoy حول مبدأ شامل نقاطی است که تابع

در برخی از آن‌ها مثبت (بیشتر از $f(0,0)=0$) و

" " " " منفی (کمتر ") است.

لذا $(0,0)$ یک نقطه زینی f است.

Partial Derivatives4

Wednesday, May 11, 2022 10:00 AM

$$z = xy - x - y$$

$$\begin{cases} f_x = y - 1 = 0 \\ f_y = x - 1 = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 1 \end{cases}$$ نقطة بحرانية

$$\Delta = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1 < 0$$

$$\Rightarrow (1,1)$$ نقطة زينية

$$f(x,y) = xy\, e^{-(x^2+y^2)/2}$$

$$f_x(x,y) = y e^{-(x^2+y^2)/2} - \cancel{x}^2 y e^{-(x^2+y^2)/2}$$

$$= y(1-x^2)\, e^{-(x^2+y^2)/2}$$

$$f_y = x(1-y^2)\, e^{-(x^2+y^2)/2}$$

$$f_{xx} = xy(x^2-3)\, e^{-(x^2+y^2)/3}$$

$$f_{xy} = (1-x^2)(1-y^2)\, e^{-(x^2+y^2)/3}$$

$$f_{yy} = xy(y^2-3)\, e^{-(x^2+y^2)/3}$$

$$\begin{cases} f_x(x,y) = 0 \\ f_y(x,y) = 0 \end{cases} \Rightarrow \begin{cases} y(1-x^2) = 0 \Rightarrow \begin{cases} x = \pm 1 \\ y = 0 \end{cases} \\ x(1-y^2) = 0 \Rightarrow \begin{cases} x = 0 \\ y = \pm 1 \end{cases} \end{cases}$$

$(1,1), (1,-1), (-1,1), (-1,-1), (0,0)$

$$\Delta(0,0)=\begin{vmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{xy}(0,0) & f_{yy}(0,0) \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1<0$$

$\Rightarrow (0,0):$ نقطه زینی

به طور مشابه: $(1,1),\ (-1,-1)$ ماکزیمم نسبی

$(1,-1),\ (-1,1)$ مینیمم نسبی

$$f(x,y,z)=x^2y+y^2z+z^2-2x$$

$$\begin{cases} f_x=2xy-2=0 \\ f_y=x^2+2yz=0 \\ f_z=y^2+2z=0 \end{cases} \Rightarrow \begin{cases} 2y^{3/2}y-2=0 \Rightarrow y=1 \\ x^2=-2yz=y^3 \Rightarrow x=1 \\ z=\dfrac{-y^2}{2} \Rightarrow z=-\tfrac{1}{2} \end{cases}$$

$\alpha=(1,1,-\tfrac{1}{2})$

$$H(X)=\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{vmatrix} = \begin{vmatrix} 2y & 2x & 0 \\ 2x & 2z & 2y \\ 0 & 2y & 2 \end{vmatrix}$$

$$H(\alpha)=\begin{vmatrix} 2 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 2 & 2 \end{vmatrix}$$

$2>0,\quad \begin{vmatrix} 2 & 2 \\ 2 & -1 \end{vmatrix} = -2-4=-6<0,\quad \begin{vmatrix} 2 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 2 & 2 \end{vmatrix} = -20<0$

$H(\alpha)$ نامعین $\Rightarrow \alpha$ نقطه زینی

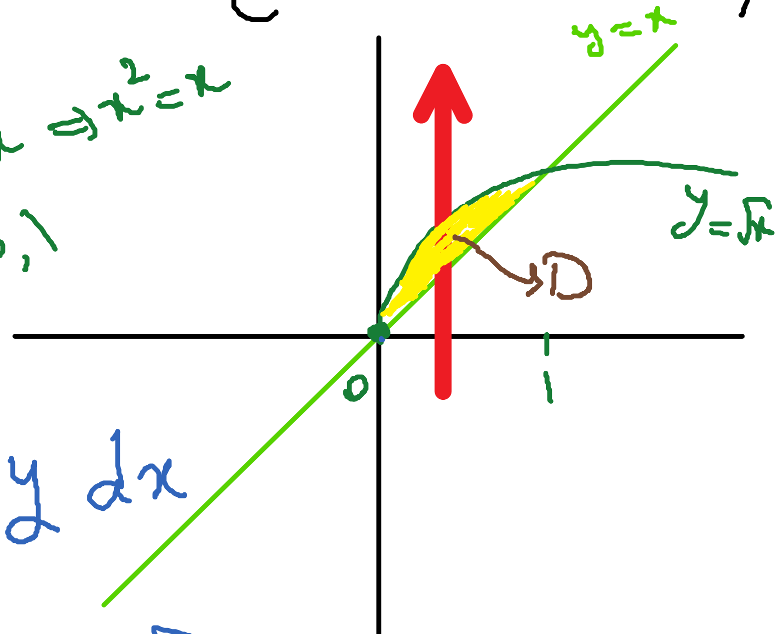
Multiple Integrals1

Monday, May 16, 2022   2:44 PM

Ex. P59 (112 file)

$$I = \iint_D (4xy - x^2)\, dA, \quad D = \{(x,y) \mid 0 \le x \le 1,\ x \le y \le \sqrt{x}\}$$

$$\left.\begin{array}{l} y = x \\ y = \sqrt{x} \end{array}\right\} \Rightarrow x = \sqrt{x} \Rightarrow x^2 = x \Rightarrow x = 0, 1$$



$$I = \int_0^1 \int_x^{\sqrt{x}} (4xy - x^2)\, dy\, dx$$

$$= \int_0^1 \left(4x \frac{y^2}{2} - x^2 y\right)\Big|_{y=x}^{y=\sqrt{x}} dx$$

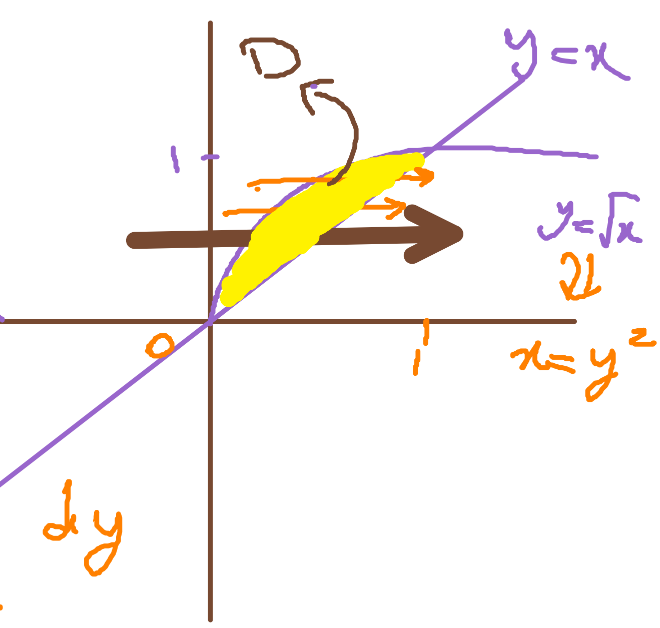
$$= \int_0^1 \left[\left(2x(\sqrt{x})^2 - x^2\sqrt{x}\right) - \left(2x\, x^2 - x^2 \cdot x\right)\right] dx$$

$$= \cdots = \int_0^1 (2x^2 - x^2\sqrt{x} - x^3)\, dx \qquad \left(x^2\sqrt{x} = x^{5/2}\right)$$

$$= 2\frac{x^3}{3} - \frac{2}{7} x^{7/2} - \frac{1}{4} x^4 \Big|_0^1 = 11/84$$

$$I = \iint_D (4xy - x^2)\, dA$$

$$= \int_0^1 \int_{y^2}^{y} (4xy - x^2)\, dx\, dy$$



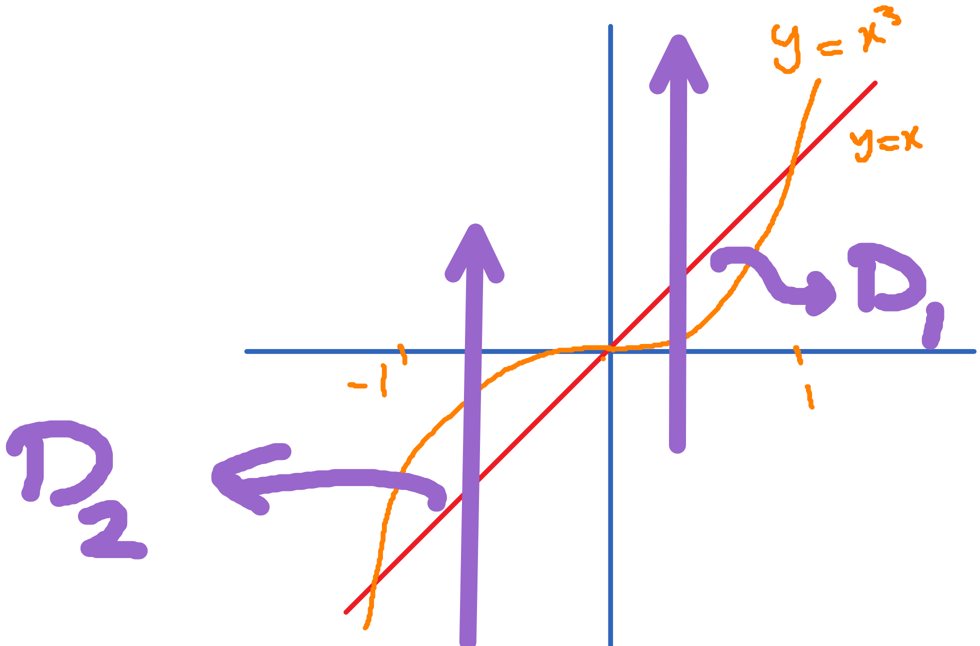
$$= \int_0^1 \left(4y \frac{x^2}{2} - \frac{1}{3} x^3\right)\Big|_{x=y^2}^{x=y} dy$$

$$= \cdots = \int_0^1 \left(\frac{5}{3}y^3 - 2y^5 + \frac{y^6}{3}\right) dy$$

$$= \frac{5}{3}\frac{y^4}{4} - 2\frac{y^6}{6} + \frac{1}{3}\frac{y^7}{7}\Bigg|_{y=0}^{y=1} = \cdots = 11/84$$

(c) $\iint_D xy\,dA$, $\quad D = D_1 \cup D_2$

$D_1: \begin{cases} 0 \le x \le 1 \\ x^3 \le y \le x \end{cases}$ $\quad D_2: \begin{cases} -1 \le x \le 0 \\ x \le y \le x^3 \end{cases}$



$$\iint_D xy\,dA = \iint_{D_1} xy\,dA + \iint_{D_2} xy\,dA$$

$$= \int_0^1 \int_{x^3}^{x} xy\,dy\,dx + \int_{-1}^0 \int_x^{x^3} xy\,dy\,dx$$

$$= \int_0^1 x\frac{y^2}{2}\Bigg|_{x^3}^{x} dx + \int_{-1}^0 x\frac{y^2}{2}\Bigg|_x^{x^3} dx$$

$$= \int_0^1 x\left(\frac{x^2}{2} - \frac{x^6}{2}\right) dx + \int_{-1}^0 x\left(\frac{x^6}{2} - \frac{x^2}{2}\right) dx$$

$$= \int_0^1 \frac{1}{2}(x^3 - x^7)\,dx + \frac{1}{2}\int_{-1}^0 (x^7 - x^3)\,dx$$

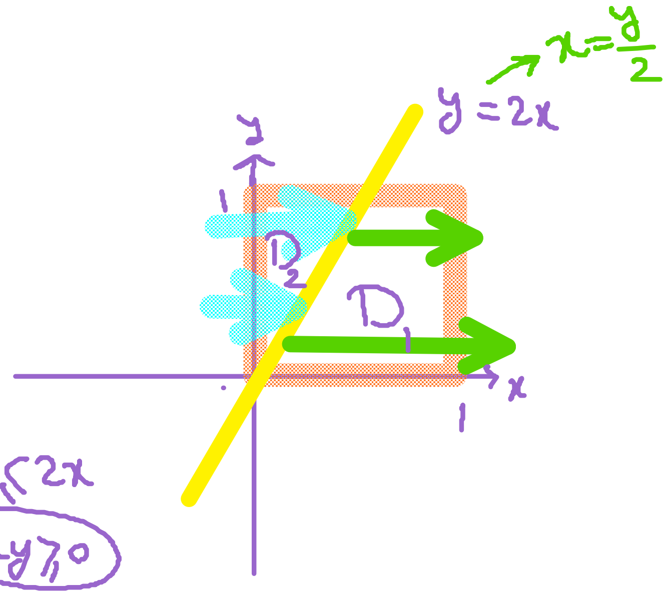
$$= \frac{1}{2}\left(\frac{x^4}{4} - \frac{x^8}{8}\right)\Big|_0^1 + \frac{1}{2}\left(\frac{x^8}{8} - \frac{x^4}{4}\right)\Big|_{-1}^0$$

$$= \ldots$$

P112 file

(d) $I=\iint_D |2x-y|\,dy\,dx$

$D=[0,1]\times[0,1]$



$y\leq 2x$

$$f(x,y)=\begin{cases}2x-y, & 2x-y\geq 0\\ y-2x, & y\geq 2x\end{cases}$$

$$I=\iint_{D_1}(2x-y)\,dA+\iint_{D_2}(y-2x)\,dA$$

$$=\int_0^1\int_{\frac{y}{2}}^1(2x-y)\,dx\,dy+\int_0^1\int_0^{\frac{y}{2}}(y-2x)\,dx\,dy$$

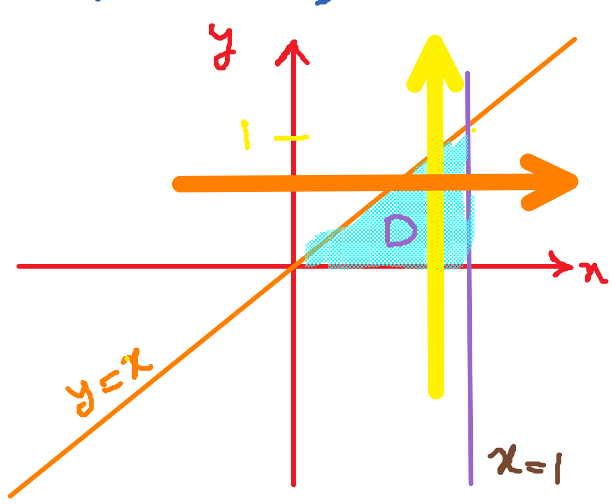
$$=\int_0^1\left[\cancel{2}\frac{x^2}{\cancel{2}}-yx\right]_{x=\frac{y}{2}}^{x=1}dy+\int_0^1\left[yx-\cancel{2}\frac{x^2}{\cancel{2}}\right]_{x=0}^{x=\frac{y}{2}}dy$$

$$=\int_0^1\Big[(1-y)-\Big(\underbrace{\frac{y^2}{4}-\frac{y^2}{2}}_{-y^2/4}\Big)\Big]dy+\int_0^1\Big[\Big(\underbrace{\frac{y^2}{2}-\frac{y^2}{4}}_{y^2/4}\Big)-0\Big]dy$$

$$=\left[y-\frac{y^2}{2}+\frac{1}{4}\frac{y^3}{3}\right]_0^1+\frac{1}{4}\,\frac{y^3}{3}\Big|_0^1=\cdots$$

(e) $I=\iint_D \frac{\sin x}{x}\,dA$

D: مثلثی در صفحه xy که به محور x ها، خطوط $y=x$ و $x=1$ محدود شده است.



$$I=\int_0^1\int_0^x\frac{\sin x}{x}\,dy\,dx$$

$$=\int_0^1\frac{\sin x}{x}\int_0^x dy\,dx$$

$$=\int_0^1\frac{\sin x}{x}\,y\Big|_0^x\,dx$$

$x=1$

$$I=\int_0^1\int_y^1 \frac{\sin x}{x}\,dx\,dy$$

انتگرال بالا را به دلیل اینکه تابع اولیه برای $\frac{\sin x}{x}$ وجود ندارد، نمی‌توان محاسبه کرد. بنابراین ناچار به تعویض حدود انتگرال هستیم

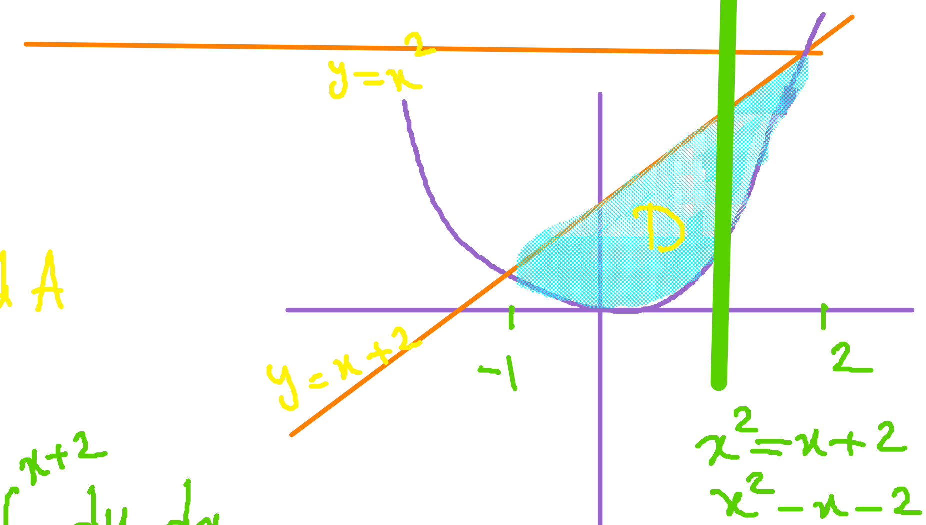
$$=\int_0^1 \frac{\sin x}{x}\, y\Big|_0^x dx$$

$$=\int_0^1 \frac{\sin x}{\cancel{x}}(\cancel{x}-0)\,dx$$

$$=-\cos x\Big|_0^1$$

$$=1-\cos(1).$$

Ex, P113



$$\begin{cases} y=x^2 \\ y=x+2 \end{cases}$$

$$x^2=x+2$$

$$x^2-x-2=0$$

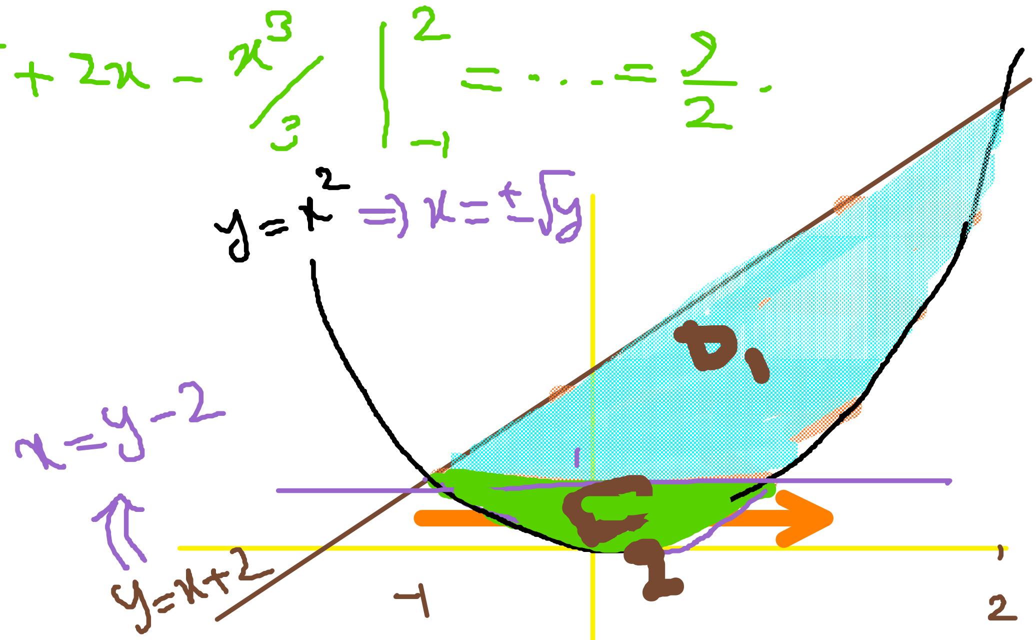
$$x=-1,\ x=2$$

$$A_D=\iint_D dA$$

$$=\int_{-1}^{2}\int_{x^2}^{x+2} dy\,dx$$

$$=\int_{-1}^{2} y\Big|_{x^2}^{x+2} dx=\int_{-1}^{2}(x+2-x^2)\,dx$$

$$=\tfrac{1}{2}x^2+2x-\tfrac{x^3}{3}\Big|_{-1}^{2}=\dots=\frac{9}{2}.$$

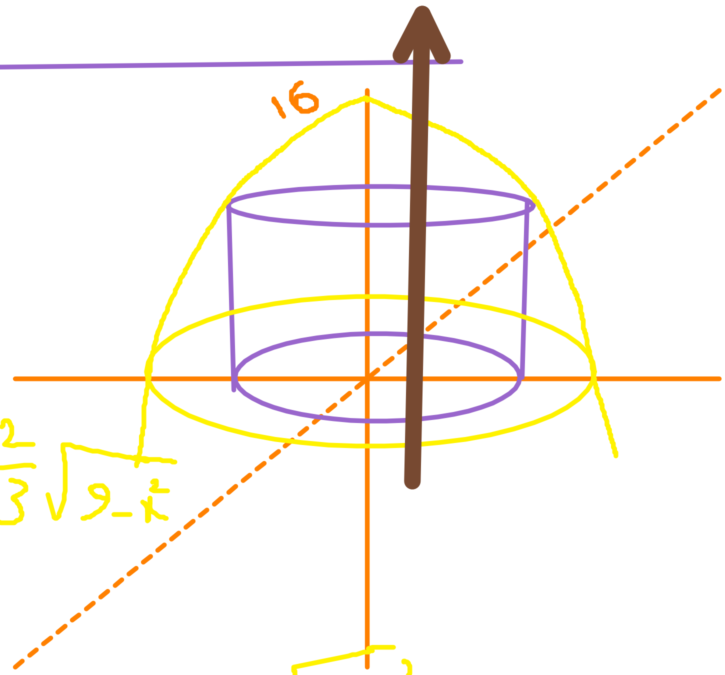


$$A_D=\iint_D dx\,dy=\iint_{D_1} dx\,dy+\iint_{D_2} dx\,dy$$

$$D \quad D \quad D_1 \quad D_2$$

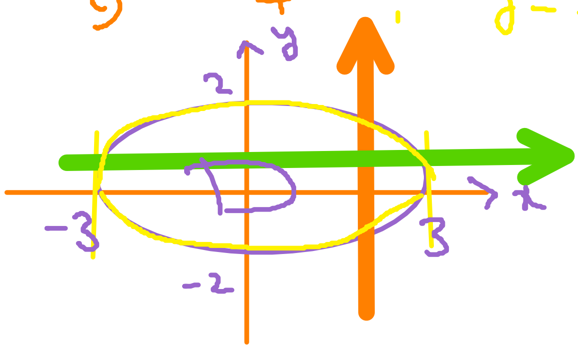
$$= \int_{-1}^{4}\int_{y-2}^{\sqrt{y}} dx\,dy + \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy = \cdots = \frac{9}{2}$$

Ex. P114



$$z = 16 - x^2 - y^2$$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 \Rightarrow y = \pm\frac{2}{3}\sqrt{9-x^2}$$



$$V = \iint_D f(x,y)\,dA = \int_{-3}^{3}\int_{-2/3\sqrt{9-x^2}}^{2/3\sqrt{9-x^2}} (16 - x^2 - y^2)\,dy\,dx$$

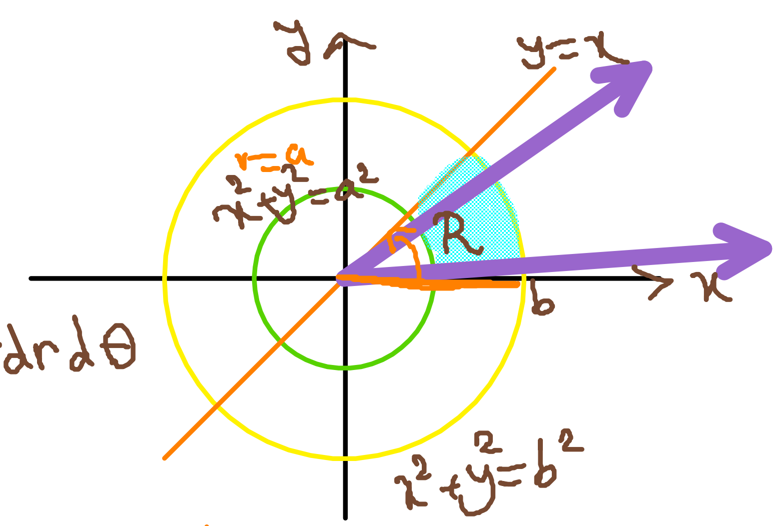
$$= \int_{-2}^{2}\int_{-\frac{3}{2}\sqrt{4-y^2}}^{3/2\sqrt{4-y^2}} (16 - x^2 - y^2)\,dx\,dy$$

ex. P119

$$I = \iint_R y^2/x^2\,dA$$

$$R = \{(x,y) \mid 0 < a^2 < x^2+y^2 < b^2\}$$

في الربع الأول وتحت الخط y=x



$$I = \int_0^{\pi/4}\int_a^b \frac{\cancel{r^2}\sin^2\theta}{\cancel{r^2}\cos^2\theta}\,r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\int_a^b \tan^2\theta\; r\,dr\,d\theta$$

$y = x \Rightarrow \tan\theta = \frac{y}{x} = 1$

$$= \int_0^{4} \int_a \tan\theta \; r\,dr\,d\theta$$

$$y = x \Rightarrow \tan\theta = \frac{y}{x} = 1$$
$$\Rightarrow \theta = \pi/4$$

$$= \int_0^{\frac{\pi}{4}} \tan^2\theta \int_a^b r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \tan^2\theta \; \frac{r^2}{2}\Big|_a^b \, d\theta$$

$$= \frac{b^2 - a^2}{2} \int_0^{\frac{\pi}{4}} (\sec^2\theta - 1)\,d\theta$$

$$= \frac{b^2 - a^2}{2} (\tan\theta - \theta)\Big|_0^{\frac{\pi}{4}} = \cdots = \frac{4-\pi}{8}(b^2 - a^2).$$

Multiple Integrals3

Wednesday, May 25, 2022 10:06 AM

Ex. P121 file

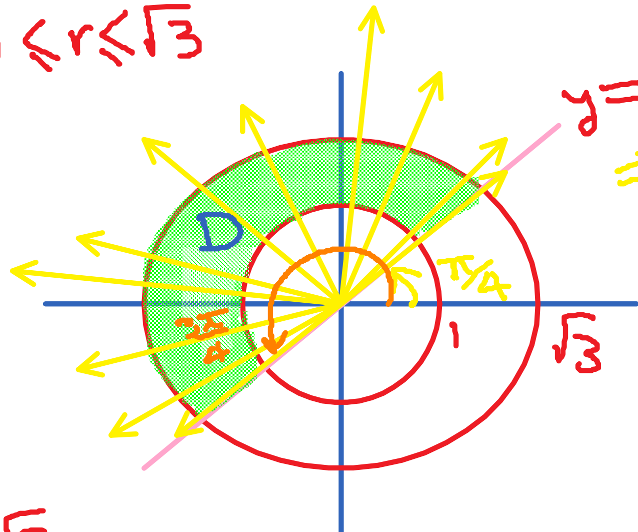
(i) $\iint_D \sin(\pi(x^2+y^2))\,dx\,dy = I$

$D = \{(x,y) \mid 1 \leq x^2+y^2 \leq 3,\ y \geq x\}$

$1 \leq r \leq \sqrt{3}$

$y = x \Rightarrow \tan\theta = \frac{y}{x} = \frac{x}{x} = 1$

$\Rightarrow \theta = \frac{\pi}{4}$



$$I = \frac{1}{2\pi}\int_{\pi/4}^{3\pi/4}\left(\int_1^{\sqrt{3}} 2\pi r \sin(\pi r^2)\,dr\right)d\theta$$

$u = \pi r^2$

$du = 2\pi r\,dr$

$r = 1 \Rightarrow u = \pi$

$r = \sqrt{3} \Rightarrow u = 3\pi$

$$= \frac{1}{2\pi}\int_{\pi/4}^{3\pi/4}\left(\int_{\pi}^{3\pi} \sin u\,du\right)d\theta$$

$$= \frac{1}{2\pi}\int_{\pi/4}^{3\pi/4}\left(-\cos u \Big|_{\pi}^{3\pi}\right)d\theta$$

$$= \frac{1}{}\int^{3\pi/4} -(-1-(-1))\,d\theta = 0$$

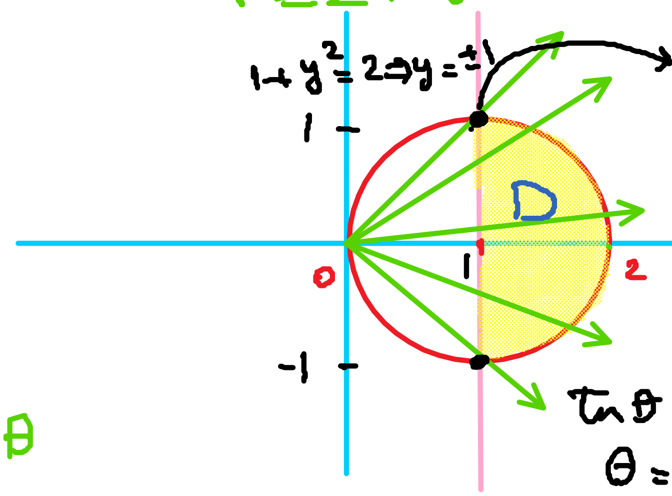
$$= \frac{1}{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{4}} -\left(-1-(-1)\right)d\theta = 0$$

(ii) $I = \iint\limits_D \frac{dx\,dy}{\sqrt{x^2+y^2}}$, $\quad D = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 \leq 2x,\ x \geq 1\}$

$x^2+y^2 = 2x \Rightarrow (x-1)^2+y^2 = 1$

$r^2 = 2r\cos\theta \Rightarrow r = 2\cos\theta$

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_{\frac{1}{\cos\theta}}^{2\cos\theta} \frac{\cancel{r}\,dr\,d\theta}{\cancel{r}}$$



$1+y^2 = 2 \Rightarrow y = \pm 1$

$\tan\theta = \frac{1}{1} = 1 \Rightarrow \theta = \frac{\pi}{4}$

$\tan\theta = \frac{-1}{1} = -1$, $\theta = -\frac{\pi}{4}$

$x = 1$

$r\cos\theta = 1 \Rightarrow r = \frac{1}{\cos\theta}$

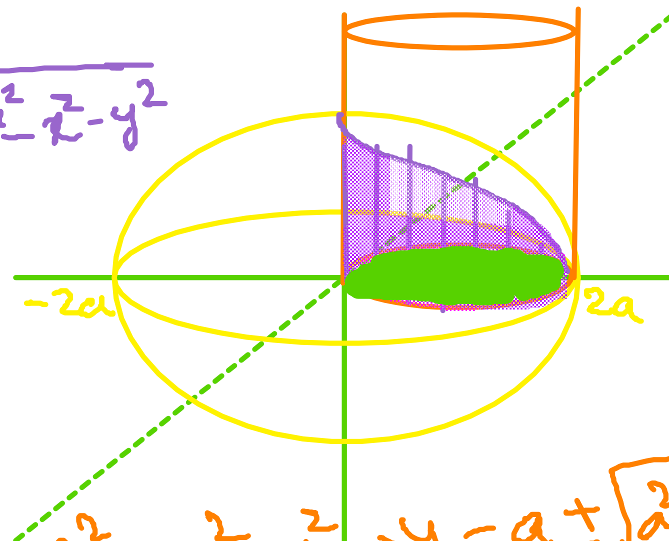
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r\Big|_{1/\cos\theta}^{2\cos\theta}\, d\theta$$

$$= \int_{-\pi/4}^{\frac{\pi}{4}} \left(2\cos\theta - \frac{1}{\cos\theta}\right)d\theta$$

$$= 2\sin\theta - \ln\left|\sec\theta+\tan\theta\right| \Big]_{-\pi/4}^{\frac{\pi}{4}} = \cdots$$

Ex. 2, P119

$$x^2+y^2+z^2=4a^2 \Rightarrow z=\sqrt{4a^2-x^2-y^2}$$

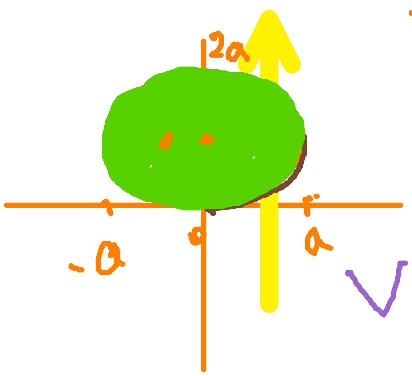


$$x^2+y^2=2ay$$

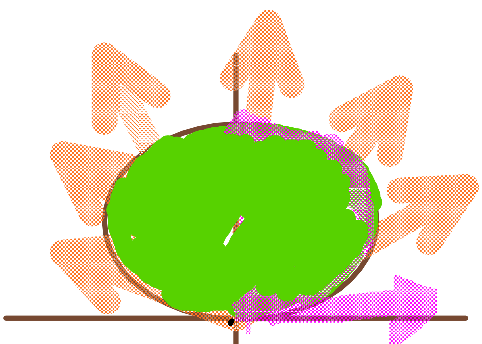
$$\underbrace{x^2+y^2-2ay+a^2}=a^2$$

$$x^2+(y-a)^2=a^2$$

$$\rightarrow (y-a)^2=a^2-x^2 \Rightarrow y=a\pm\sqrt{a^2-x^2}$$



$$V=2\iint_D z\,dA=\cancel{2}\,4\int_{\cancel{-a}\;0}^{a}\int_{a-\sqrt{a^2-x^2}}^{a+\sqrt{a^2-x^2}}\sqrt{4a^2-x^2-y^2}\,dy\,dx$$



$$=4\int_0^{\frac{\pi}{2}}\int_0^{2a\sin\theta} r\sqrt{4a^2-r^2}\,dr\,d\theta$$

$$x^2+y^2=2ay$$

$$r^2=2ar\sin\theta$$

$$r=2a\sin\theta$$

$$\boxed{\begin{array}{l} \left| \begin{array}{l} u=4a^2-r^2 \\ du=-2r\,dr \end{array}\right. \\ -\frac{1}{2}\int -2r\sqrt{4a^2-r^2}\,dr=-\frac{1}{2}\int\sqrt{u}\,du \\ =-\frac{1}{2}\frac{u^{3/2}}{3/2}=-\frac{1}{3}u\sqrt{u} \end{array}}$$

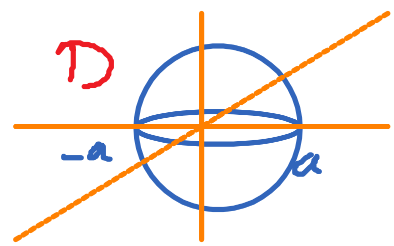
$$V = 4\int_0^{\frac{\pi}{2}} -\frac{1}{3}\left(4a^2-r^2\right)^{3/2}\Big|_0^{2a\sin\theta}\, d\theta$$

$= \ldots$

Ex. P127

(i) $\iiint (2+x-\sin z)\, dv$

$x^2+y^2+z^2 \leqslant a$



$$= 2\iiint_D dv + \iiint_D x\, dv - \iiint_D \sin z\, dv$$

$= 2V_D + 0 - 0$ → (تابع زیر انتگرال دومی فرد است و ناحیه انتگرال‌گیری متقارن)

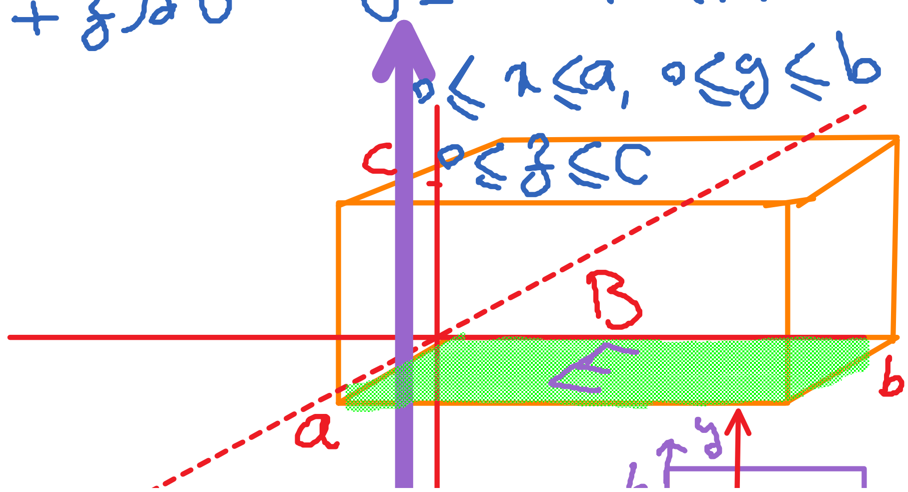
$= 2\,\frac{4}{3}\pi a^3 = \frac{8}{3}\pi a^3$

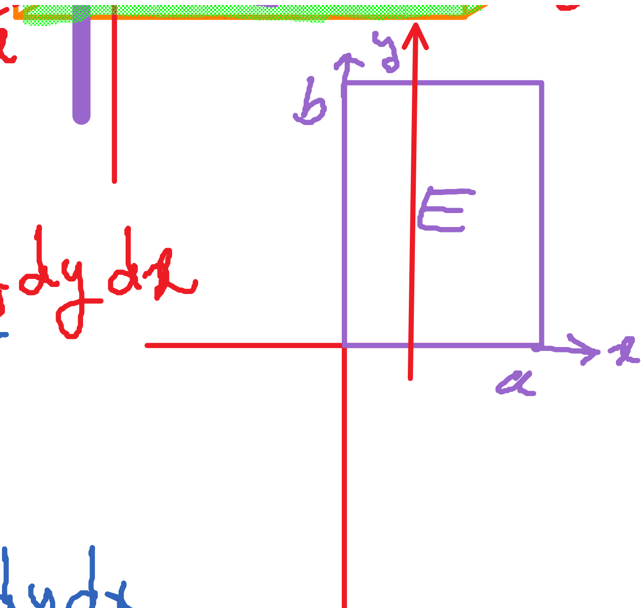
(ii) $I = \iiint_B (xy^2+z^3)\, dv$

B: جعبه مکعب مستطیل

$0 \leqslant x \leqslant a,\quad 0 \leqslant y \leqslant b$

$0 \leqslant z \leqslant c$





$$I=\int_0^a\int_0^b\int_0^c (xy^2+z^3)\,dz\,dy\,dx$$

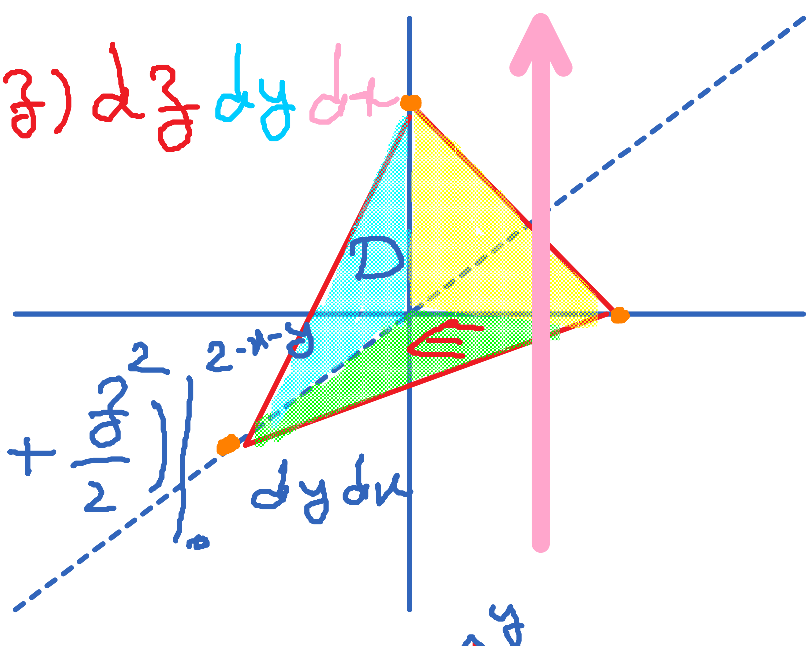
$$=\int_0^a\int_0^b\left(xy^2z+\frac{z^4}{4}\right)\Big|_0^c\,dy\,dx$$

$$=\int_0^a\int_0^b\left(cxy^2+\frac{1}{4}c^4\right)dy\,dx=\cdots$$

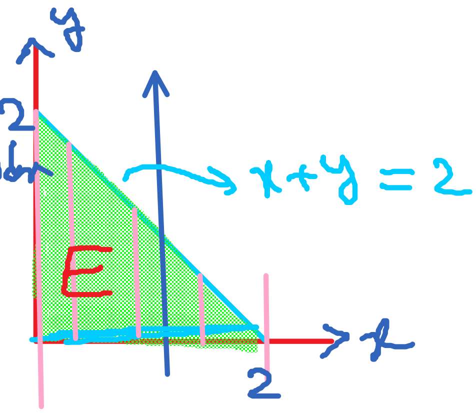
(iii) $I=\iiint\limits_D (y+z)\,dx\,dy\,dz$ ، D حجم محصور در $\frac{1}{8}$ اول فضا
توسط صفحه $x+y+z=2$

$$I=\int_0^2\int_0^{2-x}\int_0^{2-x-y}(y+z)\,dz\,dy\,dx$$

$$=\int_0^2\int_0^{2-x}\left(yz+\frac{z^2}{2}\right)\Big|_0^{2-x-y}\,dy\,dx$$



$$= \int_0^2 \int_0^{2-x} \left[y(2-x-y) + \frac{1}{2}(2-x-y)^2 \right] dy\,dx$$



$= \cdots$

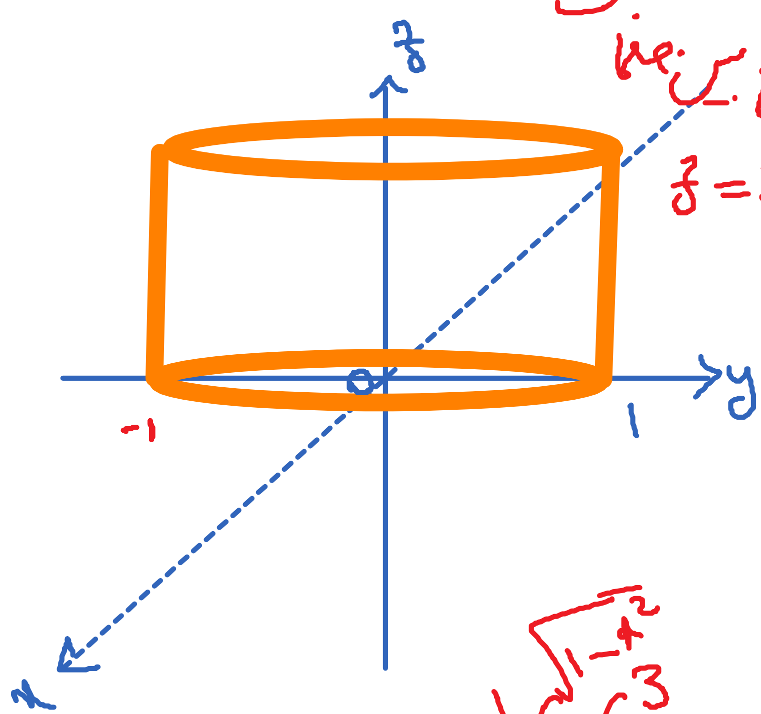
Multiple Integrals4

Monday, May 30, 2022 1:35 PM

ex. P129 file

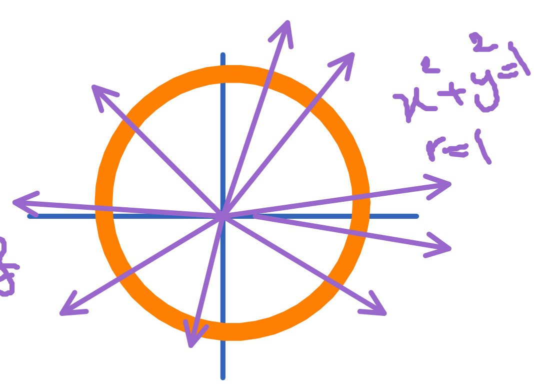
$$I=\iiint_D (x+y)\,dx\,dy\,dz$$

D: قسمتی از استوانه $x^2+y^2=1$ واقع بین $z=3$ و $z=0$



$$I=\iiint_D (x+y)\,dv=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^3 (x+y)\,dz\,dy\,dx$$

$$=\int_0^{2\pi}\int_0^1\int_0^3 r(r\cos\theta+r\sin\theta)\,dz\,dr\,d\theta$$

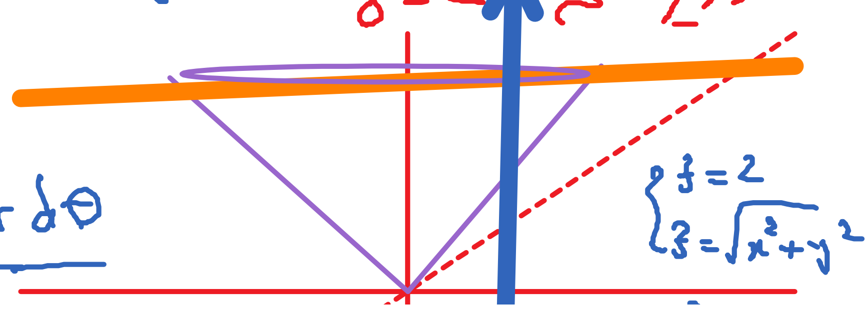


$$=\int_0^{2\pi}(\cos\theta+\sin\theta)\,d\theta\int_0^1 r^2\,dr\int_0^3 dz$$

$$=\dots$$

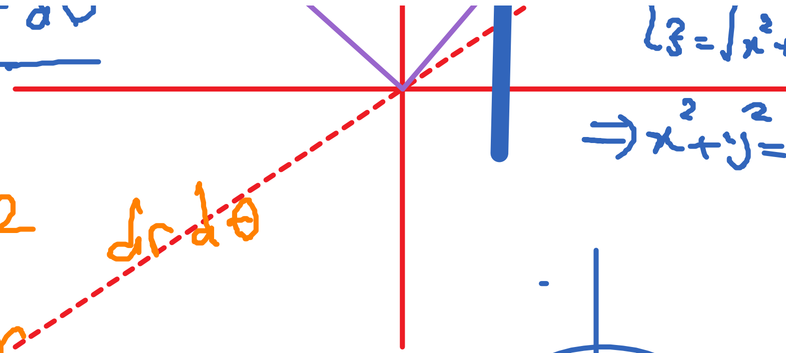
$$I=\iiint \frac{z\,dx\,dy\,dz}{\sqrt{x^2+y^2}}$$

D: ناحیه بالای مخروط $\underbrace{z=\sqrt{x^2+y^2}}_{z=r}$ و زیر صفحه $z=2$



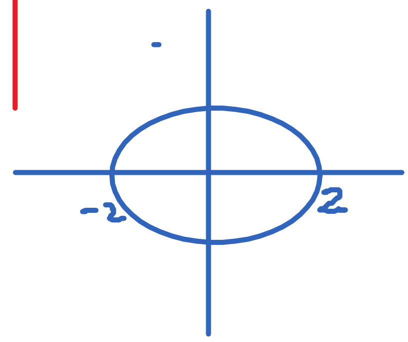
$$I=\int^{2\pi}\int^{2}\int^{2} \frac{z\,dz\,\cancel{r}\,dr\,d\theta}{\dots}$$

$$I=\int_0^{2\pi}\int_0^{2}\int_r^{2}\frac{z\,dz\,\cancel{r}\,dr\,d\theta}{\cancel{r}}$$



$z=\sqrt{x^2+y^2}$

$\Rightarrow x^2+y^2=4$



$$=\int_0^{2\pi}\int_0^{2}\frac{1}{2}z^2\Big|_r^{2}\,dr\,d\theta$$

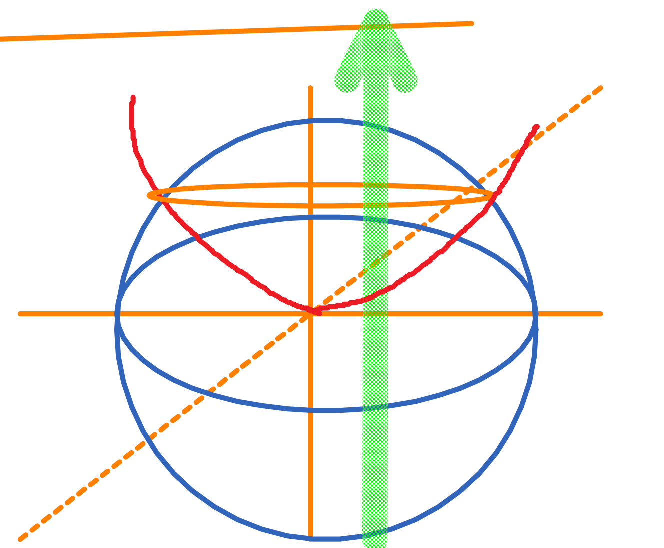
$$=\int_0^{2\pi}\int_0^{2}\frac{1}{2}(4-r^2)\,dr\,d\theta$$

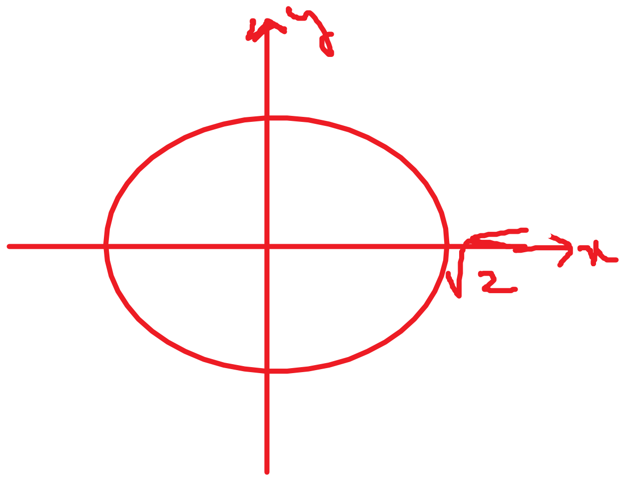
$$=2\pi\left(\frac{1}{2}\left(4r-\frac{1}{3}r^3\right)\right)\Big|_0^{2}=\frac{16}{3}\pi.$$

P130(b)

$x^2+y^2+z^2=6$

$z=x^2+y^2$





$z^2=6-x^2-y^2=(x^2+y^2)^2$

$r^4+r^2-6=0$

$(r^2-2)(r^2+3)=0$

$r=\sqrt{2}$

$$V=\iiint_D dv=\int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2}^{\sqrt{6-r^2}} r\,dz\,dr\,d\theta$$

$$=2\pi\int_0^{\sqrt{2}} r\,z\Big|_{r^2}^{\sqrt{6-r^2}}\,dr$$

$$=2\pi\int_0^{\sqrt{2}}\left(r\sqrt{6-r^2}-r^3\right)dr$$

$$= 2\pi \int_0 \left(r\sqrt{6-r^2} - r^3 \right) dr$$

$$= \frac{2\pi}{-2} \int_0 \underbrace{-2r\,dr}_{du} \sqrt{\underbrace{6-r^2}_{u}} - 2\pi \int_0^{\sqrt{2}} r^3\, dr$$

$$= -\pi\, \frac{2}{3} (6-r^2)^{3/2} \Big|_0^{\sqrt{2}} - 2\pi\, \frac{r^4}{4} \Big|_0^{\sqrt{2}}$$

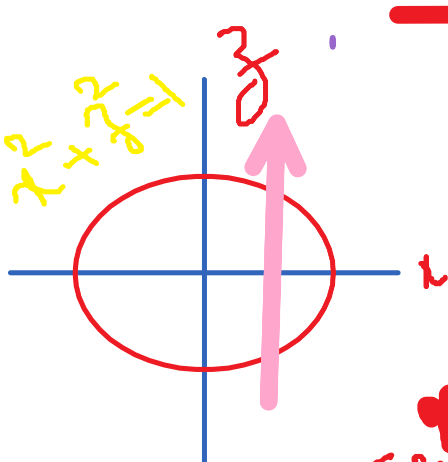
$$= \dots$$

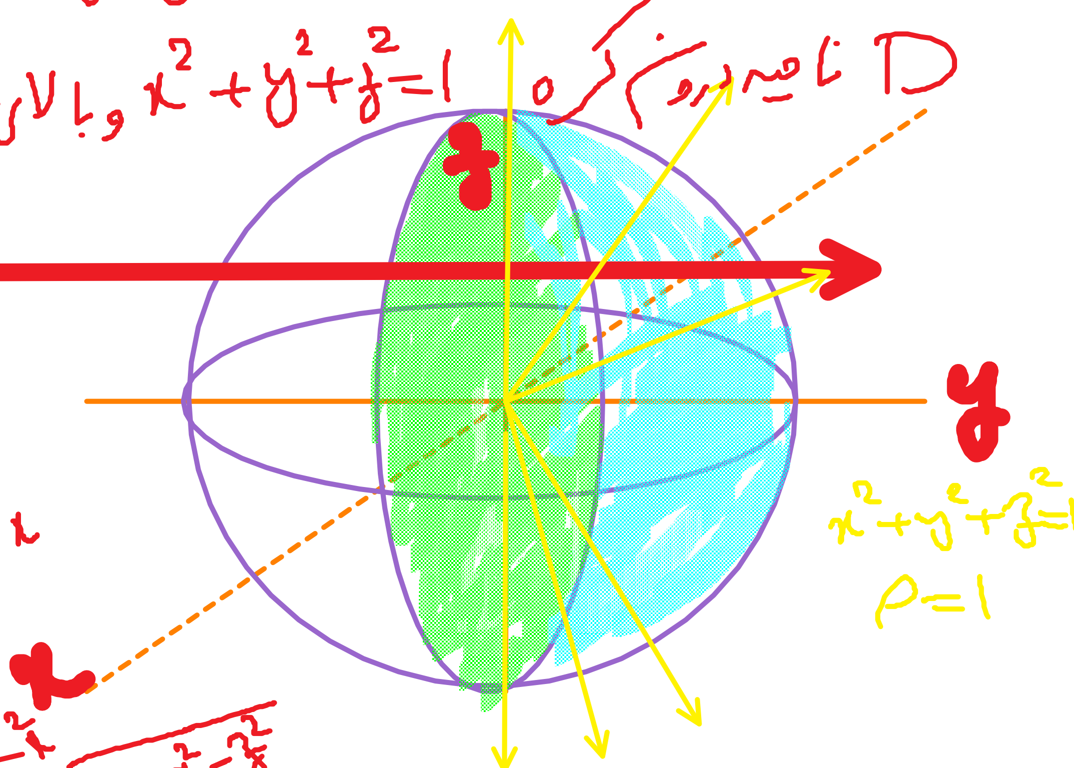
$$= \frac{2\pi}{3} \left(6\sqrt{6} - 11 \right).$$

P132

$$I = \iiint_D \sqrt{x^2+y^2+z^2}\; dx\,dy\,dz$$

D ناحیه محصور به $x^2+y^2+z^2=1$ و ... $y \geq 0$

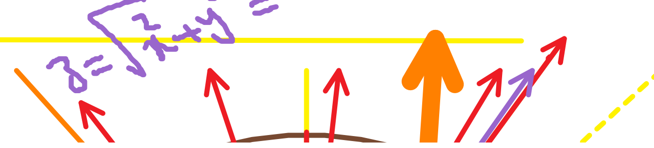




$$I = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{\sqrt{1-x^2-z^2}} \sqrt{x^2+y^2+z^2}\; dy\, dz\, dx$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{1} \underbrace{\rho\, \rho^2}_{\rho^3} S\varphi \; d\rho\, d\varphi\, d\theta$$

$z = \sqrt{x^2+y^2} = r \Rightarrow \varphi = \frac{\pi}{4}$



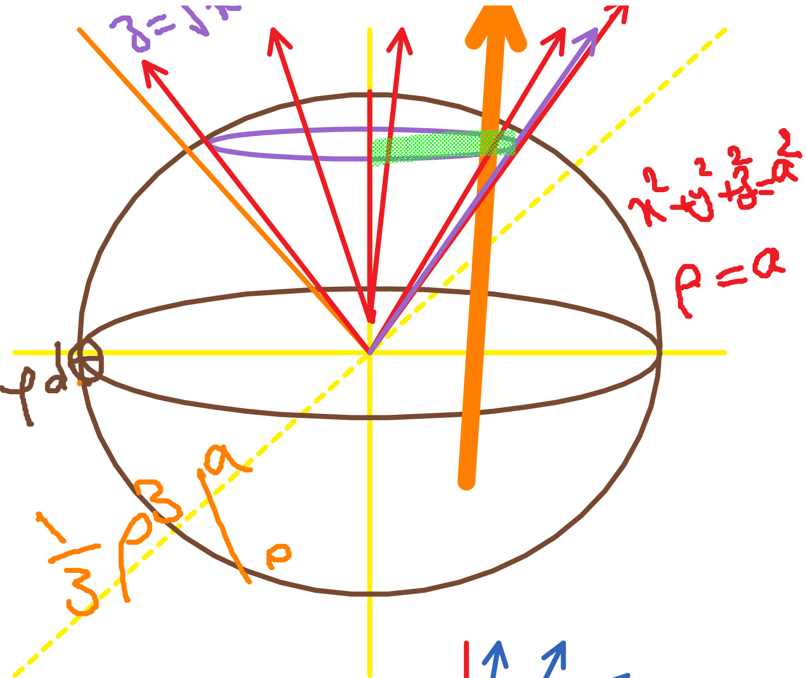
$$V=\iiint_D dv$$

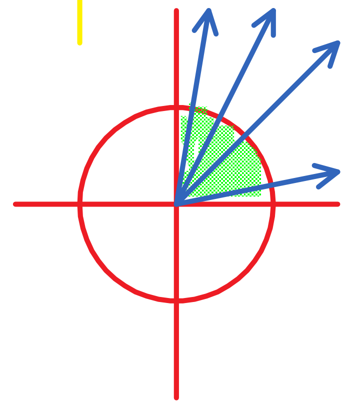
$$=4\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^a \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$=4\left(\frac{\pi}{2}-0\right)\left(-\cos\varphi\right)\Big|_0^{\frac{\pi}{4}}\ \frac{1}{3}\rho^3\Big|_0^a$$

$$=\cdots$$

$$=\frac{2\pi a^3}{3}\left(1-\frac{\sqrt{2}}{2}\right)$$





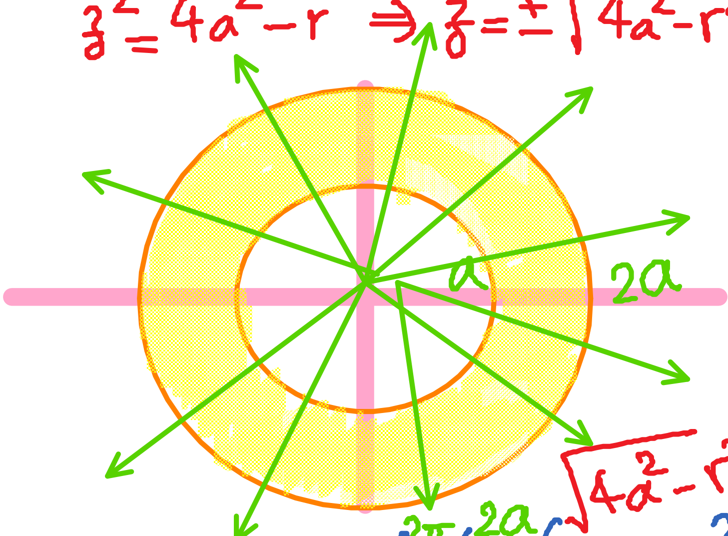
Ex. P133

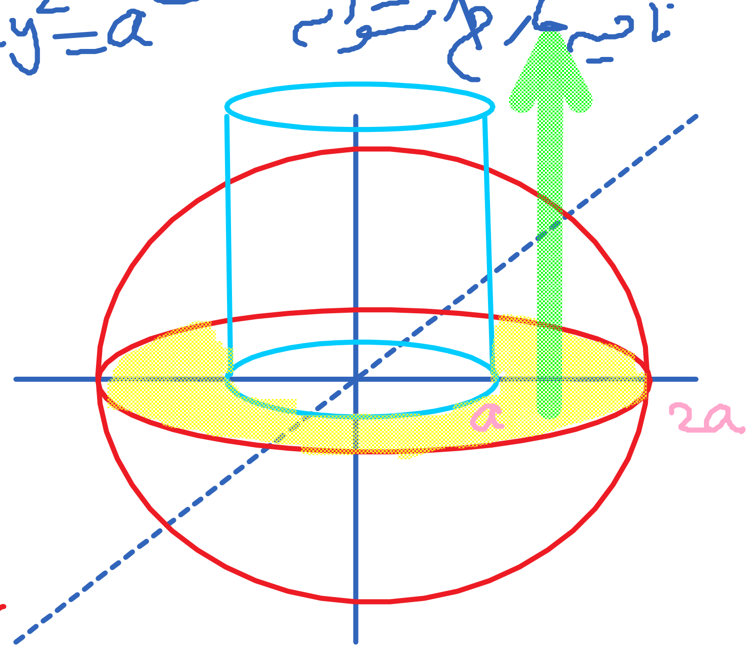
$$I=\iiint_D (x^2+y^2)\, f(x,y,z)\, dv$$

D: $x^2+y^2+z^2=4a^2$ داخل و $x^2+y^2=a^2$

ناحیه خارج استوانه

$z^2=4a^2-r^2 \Rightarrow z=\pm\sqrt{4a^2-r^2}$





$$I=2\int_0^{2\pi}\int_a^{2a}\int_0^{\sqrt{4a^2-r^2}} r^2\cdot r\, dz\, dr\, d\theta$$

$$=2(2\pi)\int^{2a} r^3\sqrt{4a^2-r^2}\, dr$$

$$= 2(2\pi)\int_{a} r\sqrt{4a - r}\; dr$$

$$= \cdots$$

$$= \frac{44\sqrt{3}}{5}\pi a^5.$$

$$I = \int_0^\infty e^{-x^2}\,dx$$

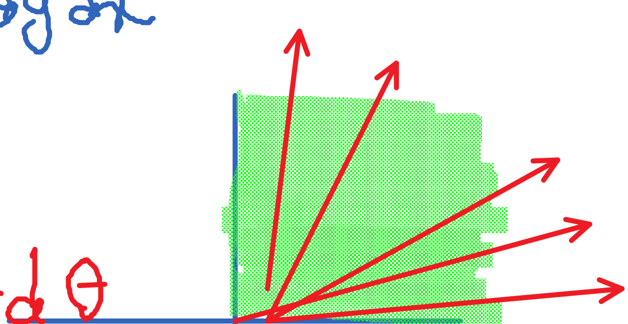
محاسبه انتگرال ناسره
با استفاده از مفاهیم انتگرال دوگانه

$$I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right)$$

$$= \int_0^\infty e^{-x^2}\int_0^\infty e^{-y^2}\,dy\,dx$$

$$= \int_0^\infty\int_0^\infty e^{-x^2}e^{-y^2}\,dy\,dx$$

$$= \int_0^\infty\int_0^\infty e^{-x^2-y^2}\,dy\,dx$$



$$= \int_0^{\frac{\pi}{2}}\frac{1}{2}\int_0^\infty 2r e^{-r^2}\,dr\,d\theta$$

$$u = r^2$$
$$du = 2r\,dr$$

$$\begin{cases} r = 0 \Rightarrow u = 0 \\ r = \infty \Rightarrow u = \infty \end{cases}$$

$$= \frac{\pi}{2} \frac{1}{2} \int_0^\infty e^{-u} \, du$$

$$= \frac{-\pi}{4} e^{-u} \Big|_0^\infty$$

$$= -\frac{\pi}{4} \left(e^{-\infty} - e^{0} \right)$$

$$= \frac{\pi}{4}$$

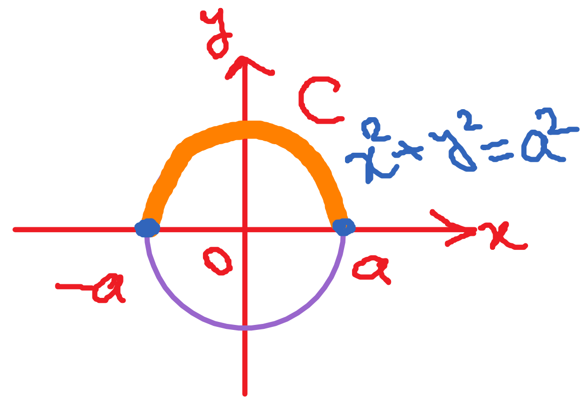
$$\Rightarrow I = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}.$$

Line integrals 1

Wednesday, June 1, 2022 10:02 AM

Ex 1, P137 file

$$M_x = \int_C y\, ds$$



$$C: \vec{r}(t) = (a\cos t, a\sin t), \quad 0 \leq t \leq \pi$$

$$\frac{d\vec{r}}{dt} = (-a\sin t, a\cos t)$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a$$

$$M_x = \int_C y\, ds = \int_0^{\pi} (a\sin t)\, a\, dt = a^2\int_0^{\pi} \sin t\, dt$$

$$= a^2(-\cos t)\Big|_0^{\pi} = -a^2(-1-1) = 2a^2.$$

$$x^2+y^2=a^2 \Rightarrow y = \pm\sqrt{a^2-x^2} \underset{y \geq 0}{\Rightarrow} y = \sqrt{a^2-x^2}$$

$$C: \vec{r}(x) = (x, \sqrt{a^2-x^2}), \quad -a \leq x \leq a$$

$$\frac{d\vec{r}}{dx} = \left(1, \frac{-x}{\sqrt{a^2-x^2}}\right)$$

$$\left|\frac{d\vec{r}}{dx}\right| = \sqrt{1 + \frac{x^2}{a^2-x^2}} = \sqrt{\frac{a^2-x^2+x^2}{a^2-x^2}}$$

$$= \frac{a}{\sqrt{a^2-x^2}} \quad (a>0)$$

$$ds = \left|\frac{d\vec{r}}{dx}\right| dx$$

$$M_x = \int y\, ds = \int_{-a}^{a} \cancel{\sqrt{a^2-x^2}}\, \frac{a}{\cancel{\sqrt{a^2-x^2}}}\, dx = a\,x\Big|_{-a}^{a}$$

$$M_x = \int_C y\,ds = \int_{-a}^{a} \cancel{\sqrt{a^2-x^2}}\ \frac{a}{\cancel{\sqrt{a^2-x^2}}}\,dx = a\,x\Big|_{-a}^{a}$$

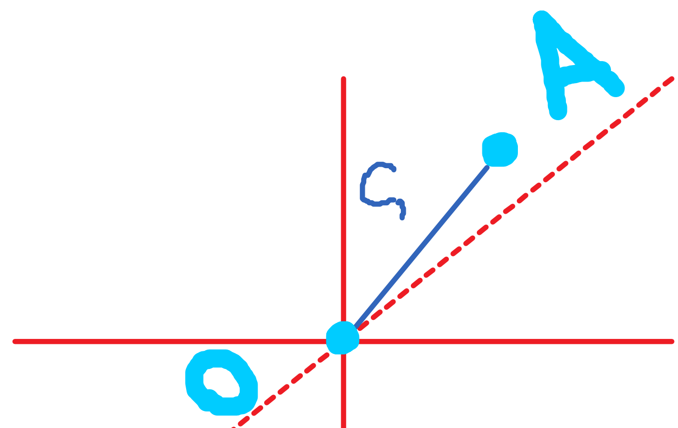
$$= a(a-(-a)) = 2a^2.$$

Ex.2, P137 file

$f(x,y,z) = 2xy + 2z - 1$

$O(0,0,0),\ A(1,1,1)$

الف) OA روی خط راست



$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\left|\frac{d\vec r}{dt}\right|dt$$

$C: \overline{OA}$ $\quad \begin{cases} x = t, \\ y = t, \\ z = t. \end{cases} \quad 0 \le t \le 1$

$$\frac{d\vec r}{dt} = (1,1,1), \quad \left|\frac{d\vec r}{dt}\right| = \sqrt{3}$$

$$\int_C f\,ds = \int_0^1 (2t^2 + 2t - 1)\sqrt{3}\,dt$$

$$= \sqrt{3}\left(\frac{2t^3}{3} + 2\frac{t^2}{2} - t\right)\Big|_0^1$$

$$= \frac{2\sqrt{3}}{3}$$

(ب) $C: \begin{cases} x = t \\ y = t^2 \end{cases}$ $\quad O \to A \quad 0 \le t \le 1$

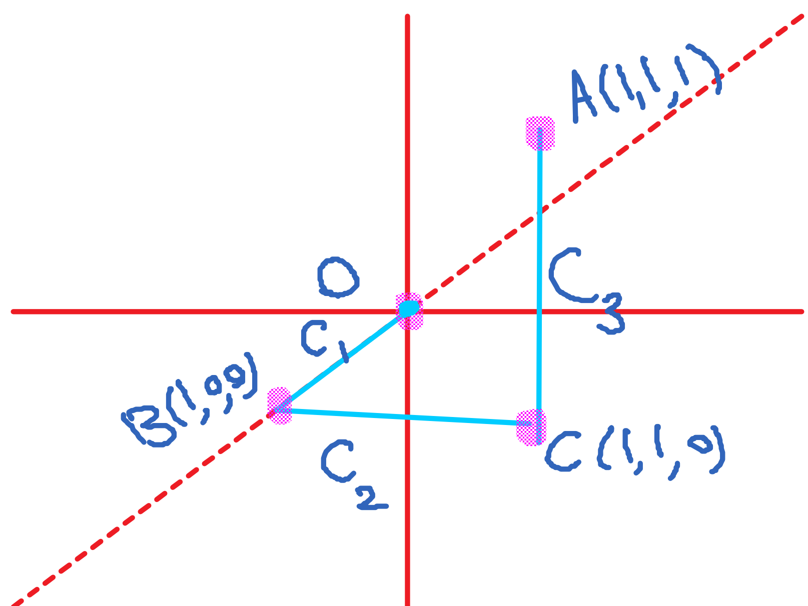
(ㄴ) $C: \begin{cases} y=t^2 \\ z=t^3 \end{cases}$

$$\frac{d\vec{r}}{dt}=(1,2t,3t^2),\quad \left|\frac{d\vec{r}}{dt}\right|=\sqrt{1+4t^2+9t^4}$$

$$\int_C f\,ds=\int_0^1 (2t^3+2t^3-1)\sqrt{1+4t^2+9t^4}\,dt$$

$$=\cdots$$

(ㄷ) $O \to B(1,0,0) \to C(1,1,0) \to A$



$$\int_C f\,ds=\int_{C_1} f\,ds+\int_{C_2} f\,ds+\int_{C_3} f\,ds$$

$O(0,0,0) \longrightarrow B(1,0,0)$

$C_1: \begin{cases} x=t, \quad 0\le t\le 1 \\ y=0 \\ z=0 \end{cases} \Rightarrow \int_{C_1} f\,ds=\int_0^1 (-1)\,dt=-1$

$$\frac{d\vec{r}}{dt}=(1,0,0),\quad \left|\frac{d\vec{r}}{dt}\right|=1$$

$B(1,0,0) \longrightarrow C(1,1,0)$

$C_2: \begin{cases} x=1 \\ y=t, \quad 0\le t\le 1 \\ z=0 \end{cases} \Rightarrow \int_C f\,ds=\int_0^1 (2t-1)\,dt$

$$C_2: \begin{cases} y = t, \; 0 \le t \le 1 \\ z = 0 \end{cases} \Rightarrow \int_{C_2} f\,ds = \int_0^1 (2t-1)\,dt$$

$$\frac{d\vec{r}}{dt} = (0,1,0), \quad \left|\frac{d\vec{r}}{dt}\right| = 1 \qquad = 2\frac{t^2}{2} - t \Big|_0^1$$

$$= 0$$

$$C(1,1,0) \longrightarrow A(1,1,1)$$

$$C_3: \begin{cases} x = 1 \\ y = 1 \\ z = t, \; 0 \le t \le 1 \end{cases} \Rightarrow \int_{C_3} f\,ds = \int_0^1 (\overbrace{2 + 2t - 1}^{2t+1})\,dt$$

$$= t^2 + t \Big|_0^1$$

$$= 2$$

$$\frac{d\vec{r}}{dt} = (0,0,1), \quad \left|\frac{d\vec{r}}{dt}\right| = \sqrt{1} = 1$$

$$\Rightarrow \int_C f\,ds = -1 + 0 + 2 = 1$$

Ex. 1, P143 file

$$\vec{F}(x,y,z) = z\vec{i} + (x+y)\vec{j} + x\vec{k} = (\underset{M(x,y,z)}{z}, \underset{N(x,y,z)}{x+y}, \underset{P(x,y,z)}{x})$$

$$O(0,0,0) \longrightarrow A(1,1,1)$$

خط راست $\overline{OA}$ الف)

$$\begin{cases} x = t, \\ y = t, \quad 0 \le t \le 1, \\ z = t, \end{cases} \qquad \frac{d\vec{r}}{dt} = (1,1,1) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

$$W = \int_C \vec{F} \cdot \vec{T}\,ds = \int_C \vec{F} \cdot d\vec{r} = \int_C (M,N,P) \cdot (dx, dy, dz)$$

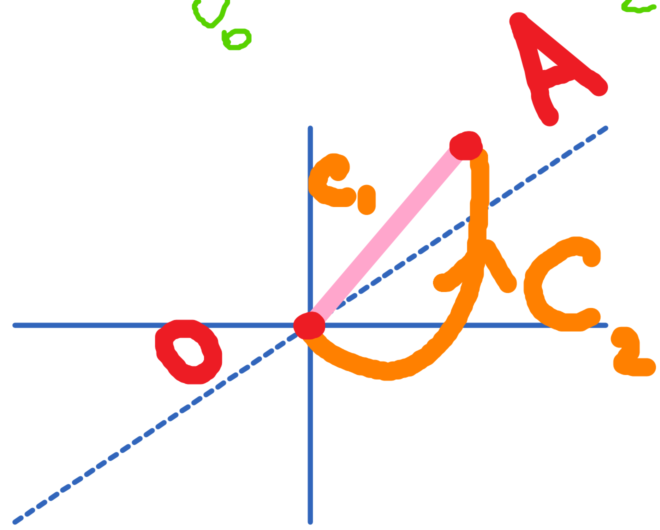
$$= \int_C M\,dx + N\,dy + P\,dz$$

$$= \int_0^1 (t, \overbrace{t+t}^{2t}, t) \cdot (1,1,1)\,dt$$

$$= \int_0^1 (t, \overbrace{t+t}^{2t}, t)\cdot(1,1,1)\,dt$$

$$= \int_0^1 (t+2t+t)\,dt = \int_0^1 4t\,dt = 4\frac{t^2}{2}\Big|_0^1$$

$$= 2$$



ب) $C: \begin{cases} x = t \\ y = t^2, \quad 0 \le t \le 1 \\ z = t^3 \end{cases}$

$$\frac{d\vec{r}}{dt} = (1, 2t, 3t^2)$$

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_0^1 (t^3, t+t^2, t)\cdot(1, 2t, 3t^2)\,dt$$

$$= \int_0^1 (\underline{t^3} + 2t^2 + \underline{2t^3} + \underline{3t^3})\,dt$$

$$= \int_0^1 (6t^3 + 2t^2)\,dt$$

$$= 6\frac{t^4}{4} + 2\frac{t^3}{3}\Big|_0^1 = \frac{3}{2} + \frac{2}{3} = \frac{13}{6}$$

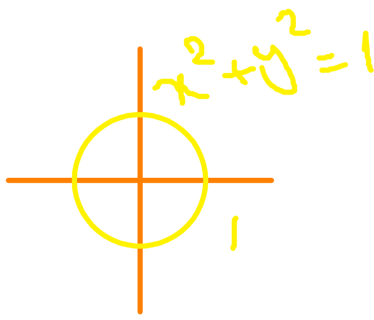
پ) $C: A \to O$

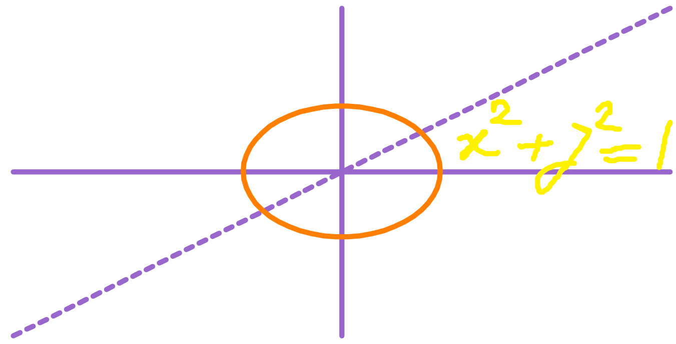
$$W = \int_C \vec{F}\cdot d\vec{r} = \int_{\overrightarrow{AO}} \vec{F}\cdot d\vec{r} = -\int_{\overrightarrow{OA}} \vec{F}\cdot d\vec{r}$$

$$= -2$$

Ex. P145 file

$$\vec{F} = (x-y)\vec{i} + x\vec{j} = (x-y,\ x)$$





$$C:\ \vec{r}(t) = (\cos t, \sin t), \quad 0 \le t \le 2\pi$$

$$\mathcal{F} = \int_C \vec{F}\cdot\vec{n}\, ds = \oint_C M\,dy - N\,dx$$

$$d\vec{r} = (-\sin t, \cos t) = (dx, dy)$$

$$\vec{F} = (x-y,\ x) = (\cos t - \sin t,\ \cos t) = (M, N)$$

$$\mathcal{F} = \int_0^{2\pi} \left[(\cos t - \sin t)\cos t - \cos t(-\sin t)\right] dt$$

$$= \int_0^{2\pi} \left(\cos^2 t - \cancel{\sin t\cos t} + \cancel{\sin t\cos t}\right) dt$$

$$= \int_0^{2\pi} \frac{1+\cos 2t}{2}\, dt$$

$$= \frac{1}{2}t + \frac{1}{2}\cdot\frac{1}{2}\sin 2t \Big|_0^{2\pi}$$

$$= \pi.$$

Line integrals 2

Monday, June 6, 2022 12:05 PM

Ex. P147, f.1e

$$\vec{F}(x,y,z) = z\vec{i} + (y+1)\vec{j} + x\vec{k}$$

$$= (\overset{M}{z}, \overset{N}{y+1}, \overset{P}{x})$$

F یک میدان پایستار (گرادیان) ?

پتانسیل f !

F پایستار iff $\begin{cases} \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \\ \dfrac{\partial M}{\partial z} = \dfrac{\partial P}{\partial x} \\ \dfrac{\partial N}{\partial z} = \dfrac{\partial P}{\partial y} \end{cases}$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 0 = \frac{\partial N}{\partial x} \\ \frac{\partial M}{\partial z} &= 1 = \frac{\partial P}{\partial x} \\ \frac{\partial N}{\partial z} &= 0 = \frac{\partial P}{\partial y} \end{aligned}\right\} \Rightarrow$$ $\vec{F}$ یک میدان پایستار (گرادیان) است

$\vec{F}$ پایستار $\Rightarrow \exists f,\ \vec{F}(x,y,z) = \vec{\nabla} f(x,y,z)$

$$(z, \underline{y+1}, \underline{x}) = (M, N, P) = (f_x, \underline{f_y}, \underline{f_z}) \quad (*)$$

$$f_x = z \overset{\int dx}{\Longrightarrow} f(x,y,z) = \int z\,dx = xz + \underline{g(y,z)} \quad (*)$$

$$\overset{(*)}{\Longrightarrow} f_y = 0 + \frac{\partial g}{\partial y} \overset{(*)}{=} y+1 \overset{\int dy}{\Longrightarrow} g(y,z) = \frac{1}{2}y^2 + y + h(z)$$

$$\Rightarrow f(x,y,z) = xz + \frac{1}{2}y^2 + y + h(z) \quad (**)$$

$$\overset{(**)}{\Longrightarrow} f_z = x + 0 + 0 + h'(z) \overset{(*)}{=} x \Rightarrow h'(z) = 0$$

$\overset{\int dz}{\Longrightarrow} h(z) = c$

$$\left|\int dz \Rightarrow h(z)=c\right.$$

$$\Rightarrow f(x,y,z)=xz+\frac{1}{2}y^2+y+c \qquad (+)$$

ب) $C:\begin{cases} x=\dfrac{2t}{1+t^2} \\ y=2t \\ z=\dfrac{1-t^2}{1+t^2}\end{cases}$ $\qquad \displaystyle\int_C \vec{F}\cdot d\vec{r}$

$A(t=0) \longrightarrow B(t=1)$

$A(0,0,0) \qquad B(1,2,0)$

$$\int \vec{F}\cdot d\vec{r}=\int (z,y+1,x)\cdot(dx,dy,dz)$$

$$dx=\frac{2(1+t^2)-(2t)^2}{(1+t^2)^2}=\frac{2-2t^2}{(1+t^2)^2}$$

$$dy=2dt$$

$$dz=\frac{-2t(1+t^2)-2t(1-t^2)}{(1+t^2)^2}=\frac{-4t}{(1+t^2)^2}$$

$$\int_C \vec{F}\cdot d\vec{r}=\int_0^1\left[\frac{1-t^2}{1+t^2}\,\frac{2-2t^2}{(1+t^2)^2}+(2t+1)2+\frac{2t}{1+t^2}\,\frac{-4t}{(1+t^2)^2}\right]dt$$

$$=\cdots$$

با توجه به اینکه $\vec{F}$ یک میدان گرادیان (الف) است بنابراین

$$\int_A^B \vec{F}\cdot d\vec{r}=\int_A^B \vec{\nabla} f\cdot d\vec{r}=f(B)-f(A)$$

$$=xz+\frac{1}{2}y^2+y+c\,\Big|_{(0,0,0)}^{(1,2,0)}$$

$$=\left(0+\frac{1}{2}2^2+2\right)-0=4$$

Ex P148 file

$$\vec{F} = (\underbrace{A x \sin(\pi y)}_{M},\ \underbrace{x^2 \cos(\pi y) + B y e^{-z}}_{N},\ \underbrace{y^2 e^{-z}}_{P})$$

A, B ? $\quad \vec{F}$ [illegible]

$\vec{F}$ [illegible] iff
$$\begin{cases} \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \Rightarrow A\pi x \cos(\pi y) = 2x\cos(\pi y) \Rightarrow A = 2/\pi \\ \dfrac{\partial N}{\partial z} = \dfrac{\partial P}{\partial y} \Rightarrow -Bye^{-z} = 2ye^{-z} \Rightarrow B = -2 \\ \dfrac{\partial M}{\partial z} = \dfrac{\partial P}{\partial x} \end{cases}$$

(i) [illegible] $\vec{F} \Rightarrow \exists f,\ \vec{F}(x,y,z) = \vec{\nabla} f(x,y,z) = (f_x, f_y, f_z)$

$$\Rightarrow \begin{cases} f_x = \dfrac{2}{\pi} x \sin(\pi y) \xRightarrow{\int dx} f(x,y,z) = \dfrac{1}{\pi} x^2 \sin(\pi y) + g(y,z) \quad (*) \\ f_y = x^2 \cos(\pi y) - 2ye^{-z} \quad (***) \\ f_z = y^2 e^{-z} \quad (☆) \end{cases}$$

$$\xRightarrow{(*)} f_z = 0 + g_z \overset{(☆)}{=} y^2 e^{-z} \xRightarrow{\int dz} g(y,z) = -y^2 e^{-z} + h(y)$$

$$\Rightarrow f(x,y,z) = \frac{1}{\pi} x^2 \sin(\pi y) - y^2 e^{-z} + h(y) \quad (**)$$

$$\Rightarrow f_y = x^2 \cos(\pi y) - 2ye^{-z} + h'(y)$$

$$\overset{(***)}{=} x^2 \cos(\pi y) - 2ye^{-z}$$

$$\Rightarrow h'(y) = 0 \Rightarrow h(y) = C$$

$$\Rightarrow f(x,y,z) = \frac{1}{\pi} x^2 \sin(\pi y) - y^2 e^{-z} + C$$

(ii)

$C: \vec{r}(t) = (\cos t, \sin t, \sin^2 t), \quad 0 \le t \le 2\pi$

$t = 0 \Rightarrow A(1,0,0)$

$t = 2\pi \Rightarrow B(1,0,0)$ $\Rightarrow$ C منحنى بسيط [illegible]

$$\oint_C \vec{F} \cdot d\vec{r} \overset{F\ [illegible]}{=} f(B) - f(A) = 0.$$

(ii) $C: \begin{cases} z = x^2 + 4y^2 \\ z = 3x - 2y \end{cases}$ $\quad A(0,0,0), \; B(1, \tfrac{1}{2}, 2)$

فصل مشترك [illegible]

$$\int_A^B \vec{F} \cdot d\vec{r} = f(B) - f(A) \quad [illegible]\ \vec{F}\ [illegible]$$

$$= \frac{1}{\pi} x^2 \sin(\pi y) - y^2 e^{-z} \Big|_{(0,0,0)}^{(1,\frac{1}{2},2)}$$

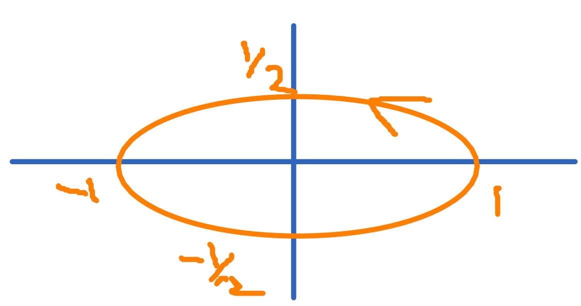
$$= \frac{1}{\pi} - \frac{1}{4e^2}$$

Ex P 149 like

$$I = \oint_C (e^x \sin y + 3y)\,dx + (e^x \cos y + 2x - 2y)\,dy$$

$4x^2 + y^2 = 4$

$x^2 + \frac{y^2}{4} = 1$



$$\vec{F} = (\underbrace{e^x \sin y + 3y}_{M}, \underbrace{e^x \cos y + 2x - 2y}_{N})$$

$\frac{\partial M}{\partial} \quad \frac{\partial N}{}$

میدان $\vec{F}$ گرادیان (بقا) iff $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

اما

$$\frac{\partial M}{\partial y} = e^{x}\cos y + 3 \neq \frac{\partial N}{\partial x} = e^{x}\cos y + 2 \Rightarrow$$

$\vec{F}$ یک میدان گرادیان نیست.

میدان F گرادیان نیست اما اگر جمله $3y$ در M، برابر $2y$ باشد آنگاه $\vec{F}$ یک میدان گرادیان است.

$$\vec{F} = \underbrace{(e^{x}\sin y + 2y,\ e^{x}\cos y + 2x - 2y)}_{\text{یک میدان گرادیان}} + (y, 0)$$

$$= \vec{\nabla} f + (y, 0)$$

$$I = \oint_C \vec{F}\cdot d\vec{r} = \oint_C \vec{\nabla} f \cdot d\vec{r} + \oint_C (y, 0)\cdot(dx, dy)$$

(انتگرال میدان گرادیان روی یک خم بسته)

$$= 0 + \oint_C y\,dx$$

$$C:\ x^2 + \frac{y^2}{4} = 1 \Rightarrow \vec{r}(t) = (\cos t, 2\sin t),\ 0 \le t \le 2\pi$$

$$d\vec{r} = (-\sin t, 2\cos t)$$

2π

$$\Rightarrow I = \int_0 (2\cos t)(-\sin t)\,dt$$

$$= -2\int_0^{2\pi} \sin^2 t\,dt = -2\int_0^{2\pi} \frac{1-\cos 2t}{2}\,dt$$

$$= -2\left.\frac{1}{2}t\right|_0^{2\pi}$$

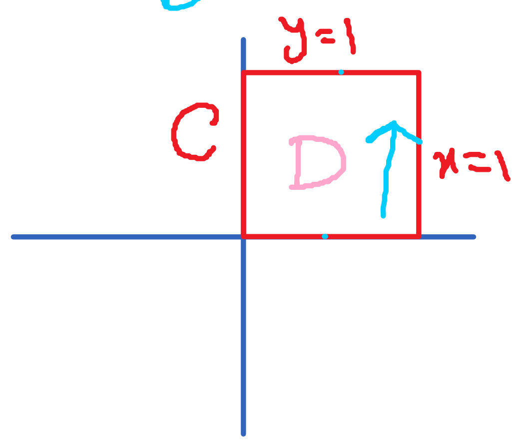
$$= -2\pi.$$

Line integrals 3

Wednesday, June 8, 2022 10:04 AM

$$\oint_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

Ex 1, P153 file



$$\oint_C \overbrace{xy}^{N}\,dy - \overbrace{y^2}^{M}\,dx$$

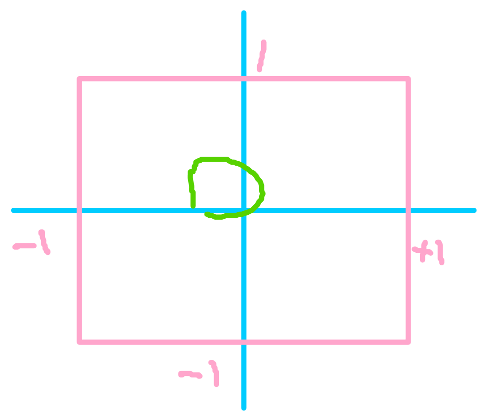
$$= \iint_D (y-(-2y))\,dA$$

$$= 3\iint_D y\,dA = 3\int_0^1\int_0^1 y\,dy\,dx = \frac{3}{2}.$$

Ex 2, P153

$$\vec{F} = x\vec{i} + y^2\vec{j}$$

$$= (x, y^2)$$



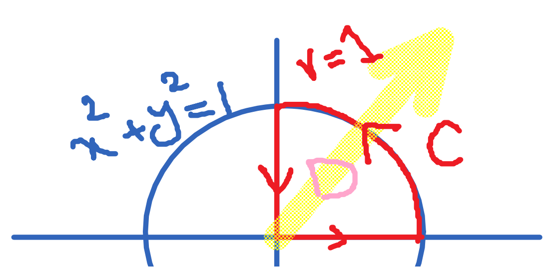
$$= \oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C M\,dy - N\,dx = \oint_C x\,dy - y^2\,dx$$

$$= \iint_D (1+2y)\,dA = \int_{-1}^{1}\int_{-1}^{1} (1+2y)\,dx\,dy$$

$$= 4$$

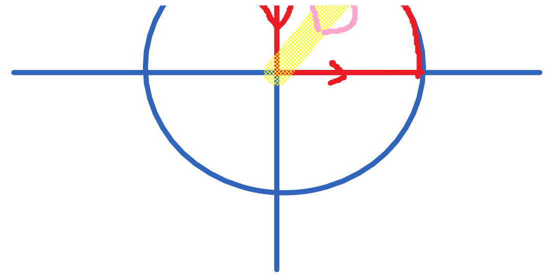
Ex 3, P153

$$I = \int_C \overbrace{(\sin x + xy)}^{M}\,dx + \overbrace{(e^y + x^2)}^{N}\,dy$$



$$\frac{\partial M}{\partial y} = x, \quad \frac{\partial N}{\partial x} = 2x$$

$\frac{\partial M}{\partial y} = x, \quad \frac{\partial N}{\partial x} = 2x$



$$I = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$= \iint_D (2x - x)\, dA = \iint_D x\, dA$$

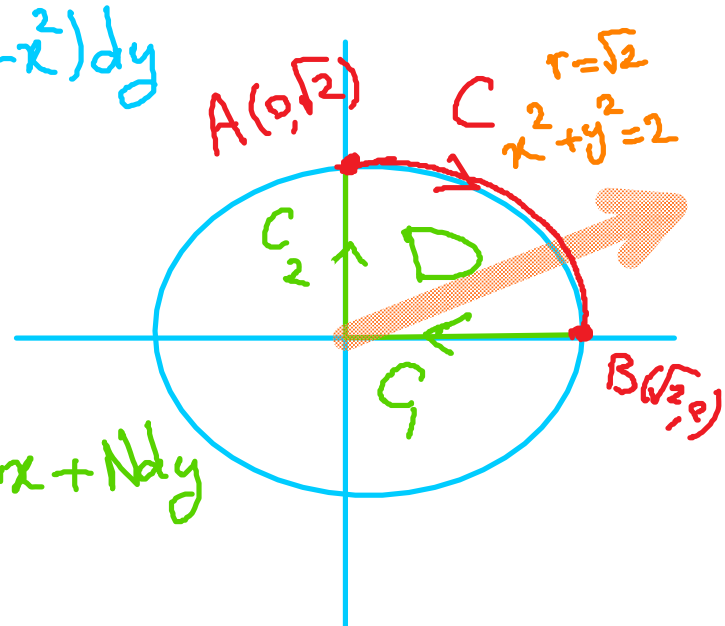
$$= \int_0^{\frac{\pi}{2}} \int_0^1 r \cos\theta \, r\, dr\, d\theta = \int_0^{\frac{\pi}{2}} \int_0^1 r^2 \cos\theta \, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos\theta \left. \frac{r^3}{3} \right|_0^1 d\theta = \frac{1}{3} \int_0^{\frac{\pi}{2}} \cos\theta \, d\theta = \frac{1}{3} \sin\theta \Big|_0^{\frac{\pi}{2}} = \frac{1}{3}.$$

Ex. iii, P 154

$M_y = 2x$
$N_x = -2x$

$$\int_C \Big(\overbrace{\frac{x}{1+x^4} + 2xy}^{M} \Big) dx + \big(\overbrace{y^2 - x^2}^{N} \big) dy$$



$$C' = C \cup C_1 \cup C_2$$

$$I = \oint_{C'} M\,dx + N\,dy = -\oint_{C'} M\,dx + N\,dy$$

گرین

$$= -\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$= -\iint_D (-2x - 2x)\, dA = 4 \iint_D x\, dA$$

$$= 4 \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}} r \cos\theta \, r\, dr\, d\theta$$

$$\int^{\frac{\pi}{2}} \int^{\sqrt{2}} r^2 \cos\theta \, dr\, d\theta = \frac{8\sqrt{2}}{}$$

$$= 4\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{2}} r^2 \cos\theta \, dr\, d\theta = \frac{8\sqrt{2}}{3}.$$

$$I = \int_{C'} M\,dx + N\,dy = \int_C \cdots + \int_{C_1} \cdots + \int_{C_2} \cdots$$

$$-C_1: \begin{cases} x = t, & 0 \leq t \leq \sqrt{2} \\ y = 0 \end{cases}$$

$$-\int_{C_1} M\,dx + N\,dy = \int_{-C_1} M\,dx + N\,dy = \int_0^{\sqrt{2}} \frac{t}{1+t^4}\,dt$$

$$= \frac{1}{2}\tan^{-1} t^2 \Big|_0^{\sqrt{2}} = \frac{1}{2}\tan^{-1}(2)$$

$$\Rightarrow \int_{C_1} M\,dx + N\,dy = -\tfrac{1}{2}\tan^{-1}(2)$$

$$C_2: \begin{cases} x = 0 \\ y = t, & 0 \leq t \leq \sqrt{2} \end{cases} \qquad \begin{cases} dx = 0 \\ dy = dt \end{cases}$$

$$\int_{C_2} M\,dx + N\,dy = \int_0^{\sqrt{2}} t^2\,dt = \frac{2}{3}\sqrt{2}$$

$$\Rightarrow \int_C M\,dx + N\,dy = I - \int_{C_1} M\,dx + N\,dy - \int_{C_2} M\,dx + N\,dy$$

$$= \frac{8\sqrt{2}}{3} + \frac{1}{2}\tan^{-1}(2) - \frac{2}{3}\sqrt{2}$$

$$= 2\sqrt{2} + \tfrac{1}{2}\tan^{-1}(2).$$